# Part 5. Computational Complexity (1)

CS 200 Algorithms and Data Structures



## Outline

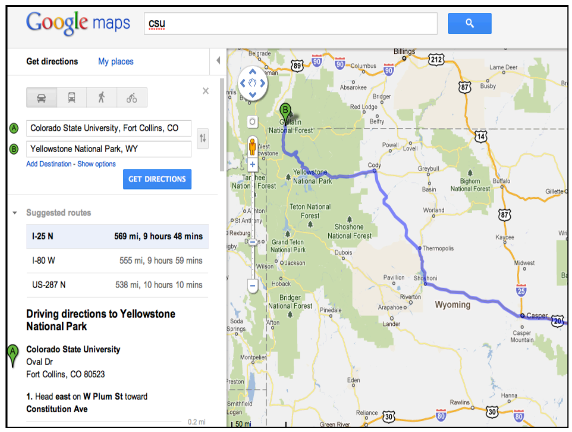
- **Measuring efficiencies of algorithms**
- Growth of functions
- Asymptotic bounds for the growth functions
- Big-O notation



## Algorithm and Computational Complexity

- An ***algorithm*** is a finite sequence of precise instructions for performing a computation for solving a problem.
- ***Computational complexity*** measures the processing time and computer memory required by the algorithm to solve problems of particular size.






## Software cost factors

- Human costs
  - Time of developers, testers, maintainers, support team, users
- Managing human costs
  - Modularity and Abstraction
  - Information hiding, good style, readability



## Software cost factors (cont'd)

- Efficiency of algorithms
  - **Time** to execute algorithms
  - **Space** required by algorithms

## Measuring the efficiency of algorithms

- We have two algorithms, `alg1` and `alg2`, that solve the same problem.
  - Our application needs a **fast** running time.
- **How do we choose** between the algorithms?

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## Example (1/3)

- An elevator is trying to move 25 packages to the first level.
- From second to sixth floor, on each of the floors, we have 5 packages *waiting* for the elevator to be transported to the first floor.
- The elevator can move 10 packages at a time.
- Initially elevator is on the first floor.

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## Example (2/3)

- **Solution 1**: Transport packages on the second floor to the first floor, and transport packages on the third floor to the first floor, etc.
- **Analysis of Solution 1**: What is the total number of levels that the elevator traveled?

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## Example (3/3)

- **Solution 2**:
  - Pick up packages on the third floor, and stop and pick up packages at the second floor and unload them on the first.
  - Pick up packages on the fifth floor and pick up packages on the fourth floor and unload them on the first floor.
  - Pick up packages on the sixth floor and move them to the first floor.
- **Analysis of Solution 1**: What is the total number of levels travelled?

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## Comparing Solution 1 and 2 for n levels

- Solution 1: $2(1+2+3+\ldots+n)$
- Solution 2:
  - $2(2+4+6+\ldots+n)$ (if $n$ is even)
  - $2(2+4+6+\ldots(n-1)+n)$ (if $n$ is odd)

| | n = 1 | n = 2 | n = 3 | n = 4 | n = 5 | n = 6 |
|---|---|---|---|---|---|---|
| Solution 1 | 2 | 6 | 12 | 20 | 30 | 42 |
| Solution 2 | 2 | 4 | 10 | 12 | 22 | 24 |
| Difference | 0 | 2 | 2 | 8 | 8 | 18 |

- In this comparison what did we ignore?

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## Measuring the efficiency of algorithms

- Implement the two versions in Java and compare their running times
- Issues with this approach:
  - *How are the algorithms coded*? We want to compare the algorithms, not the implementations.
  - *What computer should we use*? Choice of operations could favor one implementation over another.
  - *What data should we use*? Choice of data could favor one algorithm over another

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## Measuring the efficiency of algorithms

- Objective: **Analyze** algorithms independent of specific implementations, hardware, or data
- Observation: An algorithm's execution time is related to the number of operations it executes
- Solution: *Count* the number of **significant** operations the algorithm will perform for the given problem size

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## Examples

- Copying an array with $n$ elements requires ____ invocations of operations

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## Example

- Finding the maximum element in a finite sequence

```
public int max (in: array of positive integers a[])
  int max=-1;
   for (int i = 0; i < size_of_array; i++){
        if ( max < a[i] ) max = a[i];
   }
   return max;
}
```

For the input array with size of $n$ integers, `for` loop is executed **n** times.

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## Growth rates

**Algorithm A** requires $n^2 / 2$ operations to solve a problem of size $n$

**Algorithm B** requires $5n+10$ operations to solve a problem of size $n$

***Which one would you choose?***

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## Growth rates

- When we increase the size of input $n$, how the **execution time grows** for these algorithms?

| **n** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** |
|---|---|---|---|---|---|---|---|---|
| $n^2$ / 2 | 1/2 | 4/2 | 9/2 | 16/2 | 25/2 | 36/2 | 49/2 | 64/2 |
| 5n+10 | 15 | 20 | 45 | 30 | 35 | 40 | 45 | 50 |

| **n** | **50** | **100** | **1,000** | **10,000** | **…** |
|---|---|---|---|---|---|
| $n^2$ / 2 | 1250 | 5,000 | 500,000 | 50,000,000 | … |
| 5n+10 | 260 | 510 | 5,010 | 50,010 | … |

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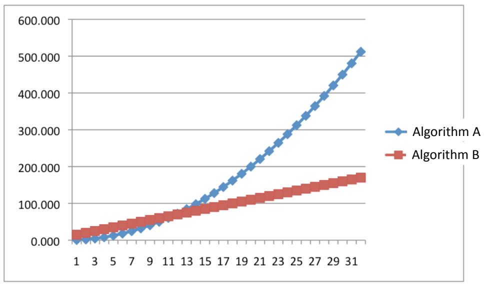
## Growth Rates

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## Growth rates

- Important to know **how quickly an algorithm's execution time grows** as a ***function of its input data size***.
- We focus on the growth rate:
  - Algorithm A requires time proportional to $n^2$
  - Algorithm B requires time proportional to $n$
  - ***B's time requirements grows more slowly than A's time requirement (for large n)***

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## Growth of functions

- Measure the **speed** of algorithm
  - Growth of number of operations (relative to the input size)
  - $n$ : input size, positive integer
  - $x$: real number

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## Asymptotic Bounds

- Big-O notation
- Big-Omega notation
- Big-Theta notation

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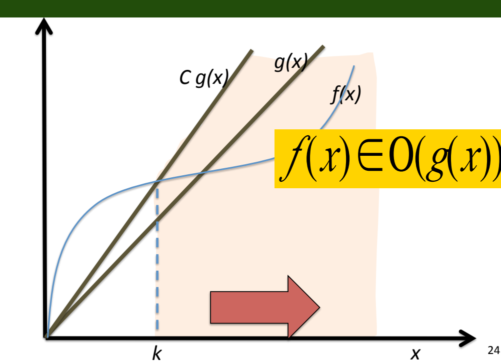


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Let $f$ and $g$ be functions from the set of integers or the set of real numbers to the set of real numbers. We say $f(x)$ is $O(g(x))$ If there are constants $C$ and $k$ such that,

$$|f(x)| \leq C|g(x)|$$

whenever $x > k$

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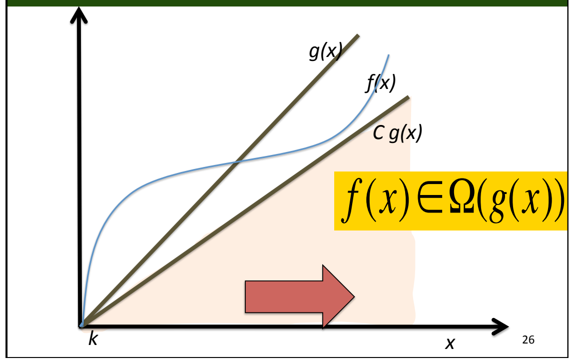


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Let $f$ and $g$ be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $f(x)$ is $\Omega(g(x))$ if there are positive constants $C$ and $k$ such that,

$$|f(x)| \geq C|g(x)|$$

whenever $x > k$

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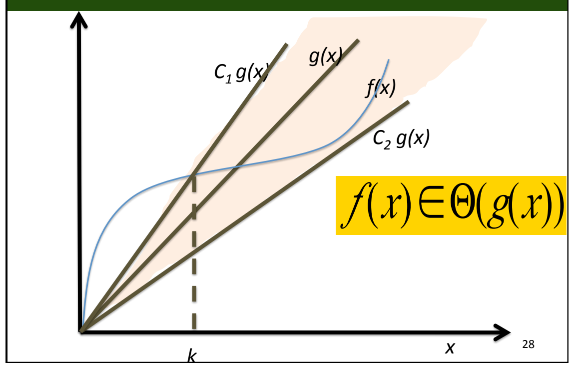


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Let $f$ and $g$ be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $f(x)$ is $\Theta(g(x))$ if $f(x)$ is $O(g(x))$ and $f(x)$ is $\Omega(g(x))$

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- **Big-O notation**

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## Order of magnitude analysis

Big O notation:
- Let $f$ and $g$ be functions from the set of integers or the set of real numbers to the set of real numbers.
  We say $f(x)$ is $O(g(x))$ . If there are constants $C$ and $k$ such that, $|f(x)| \leq C|g(x)|$ whenever $x > k$
- Focus is on the **shape** of the function
  – Ignore the multiplicative constant
- Focus is on **large** $x$
  – k allows us to ignore behavior for small $x$

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## Order of magnitude analysis

Big O notation:
- Let $f$ and $g$ be functions from the set of integers or the set of real numbers to the set of real numbers.
  We say $f(x)$ is $O(g(x))$ . If there are constants $C$ and $k$ such that, $|f(x)| \leq C|g(x)|$ whenever $x > k$
- $C$ and $k$ are witnesses to the relationship that $f(x)$ is $O(g(x))$
- If there is one pair of witnesses $(C,k)$ then there are *infinitely* many.

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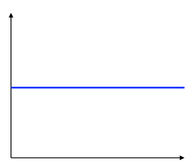
## Common Shapes: Constant

- $O(1)$



- examples?

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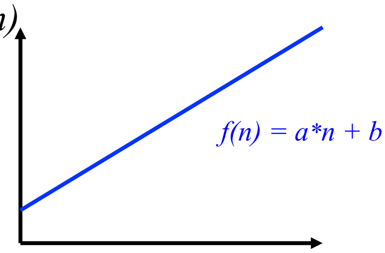
## Common Shapes: Linear

- $O(n)$

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## Linear

Example: copying an array

```
  for (int i = 0; i < a.size; i++){
A     a[i] = b[i];
  }
```

How many times should the line A be executed to finish this task?

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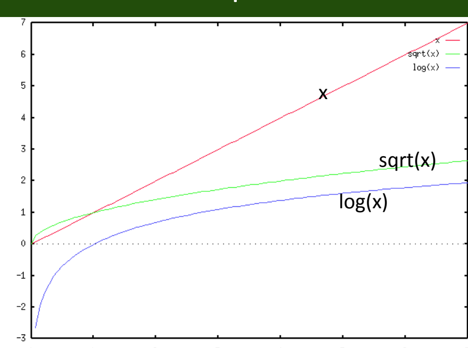
## Other Shapes: Sublinear

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## Common Shapes: logarithm

- $log_b n$ is the number $x$ such that $b^x = n$

  $2^3 = 8$ $\log_2 8 = 3$

  $2^4 = 16$ $\log_2 16 = 4$

- We usually work with base 2

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## Logarithms (cont.)

- Properties of logarithms
  - $log(x\,y) = log\,x + log\,y$
  - $log(x^a) = a\,log\,x$
  - $log_a n = log_b n \,/\, log_b a$
- logarithm is a **very slow-growing** function
- Examples of logarithmic complexity?

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## Quadratic

$O(n^2)$:

```
for (int i=0; i < n; i++){
    for (int j=0; j < n; j++) {
    }
}
```

*n* times

*n* times

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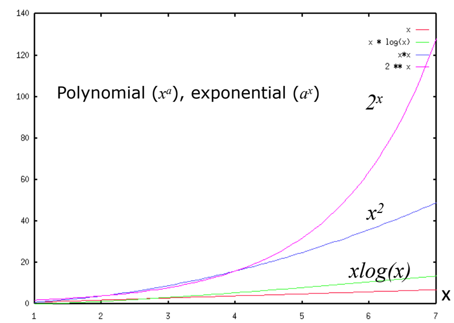
## Other Shapes: Superlinear

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## Proof

- Show that $f(x) = x^2+2x+1$ is $O(x^2)$

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## Proof

- Show that $f(x) = 7x^2$ is $O(x^3)$

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## Proof

- Show that $f(x) = x^2$ is NOT $O(x)$



## Big-O for Polynomials

Theorem: Let

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

where $a_n, a_{n-1} \ldots, a_1, a_0$ are real numbers.

Then $f(x)$ is $O(x^n)$

Example: $x^2 + 5x$ is $O(x^2)$

Proof:



## Properties of Growth-rate functions(1/3)

1. You can ignore low-order terms in an algorithm's growth-rate function.

- $O(n^3+4n^2+3n)$ it is also $O(n^3)$



## Properties of Growth-rate functions(2/3)

2. You can ignore a multiplicative constant in the high-order term of an algorithm's growth-rate function

- $O(5n^3)$, it is also $O(n^3)$



## Properties of Growth-rate functions (3/3)

3. You can combine growth-rate functions

- $O(n^2) + O(n)$, it is also $O(n^2+n)$
- Which you write as $O(n^2)$



## Combinations of Functions

- Additive Theorem:

Suppose that $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$.
Then $(f_1 + f_2)(x)$ is $O(\max(|g_1(x)|, |g_2(x)|)$.

- Multiplicative Theorem:

Suppose that $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$.
Then $(f_1 f_2)(x)$ is $O(g_1(x) g_2(x))$.

## Practical Analysis - Combinations

- Sequential
  - Big-$O$ bound: Steepest growth dominates
  - Example: copying of array, followed by binary search
    - $n + log(n)$   $O(?)$
- Embedded code
  - Big-$O$ bound multiplicative
  - Example: a for loop with $n$ iterations and a body taking $O(log\ n)$   $O(?)$

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