Part 5.
Computational Complexity (2)

CS 200 Algorithms and Data Structures

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Outline

- **Complexity of Algorithms**
- Efficiency of Searching Algorithms
- Sorting Algorithms and Their Efficiencies

2

(revisit) Properties of Growth-rate functions(1/3)

1. You can ignore low-order terms in an algorithm's growth-rate function.

- $O(n^3+4n^2+3n)$ it is also $O(n^3)$

3

(revisit) Properties of Growth-rate functions(2/3)

2. You can ignore a multiplicative constant in the high-order term of an algorithm's growth-rate function

- $O(5n^3)$, it is also $O(n^3)$

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(revisit) Properties of Growth-rate functions (3/3)

3. You can combine growth-rate functions

- $O(n^2) + O(n)$, it is also $O(n^2+n)$
- Which you write as $O(n^2)$

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Examples

- Determine whether each of these functions is $O(x^2)$
- $f(x) = 17x + 11$
- $f(x) = x^2 + 1000$
- $f(x) = xlogx$
- $f(x) = x^4/2$
- $f(x) = 2^x$

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Examples

- Is $(x^2 + 1)/(x + 1) \quad O(x)$?

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What is the Satisfied Solution?

- Algorithm should provide **correct** answer.
- Algorithm should be **efficient**.

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Demonstrating Efficiency

- Computational complexity of the algorithm
  - Time complexity
  - Space complexity
    - Analysis of the computer memory required
    - Data structures used to implement the algorithm

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Best, Average, and Worst Cases

- **Worst case**
  - Just how bad can it get:
    - The maximal number of steps
- **Average case**
  - Amount of time expected "usually"
- **Best case**
  - The smallest number of steps

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Sequential Search

0	1	3	4	5	6	7	8	9	10	11	12	13	14

- Array of n items
  - From the first one until either you find the item or reach the end of the array.
  - Best case: $O(1)$
  - Worst case: $O(n)$ (n times of comparison)
  - Average case: $O(n)$ ($n/2$ comparison)

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Binary Search (1/3)

- Searches a sorted array for a particular item by repeatedly dividing the array in half.
- Determines which half the item must be in and discards other half.
- Suppose that $n = 2^k$ for some k. (n=1,2,4,8,16,…)
  1. Inspect the middle item of size n
  2. Inspect the middle item of size $n/2$
  3. Inspect the middle item of size $n/2^2$
  4. .
  5. .
  6. .

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Binary Search (2/3)

- Dividing array in ***half*** k times.
- Worst case
  - Algorithm performs k divisions and k comparisons.
  - Since $n = 2^k$, $k = \log_2 n$
  - $O(\log_2 n)$

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Binary Search (3/3)

- What if n is not a power of 2?
- We can find the smallest k such that,

$2^{k-1} < n < 2^k$
$k - 1 < \log_2 n < k$
$k < 1 + \log_2 n < k+1$
$k = 1 + \log_2 n$ rounded down

Therefore, the algorithm is still $O(\log_2 n)$.

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Is Binary Search is more Efficient than Linear Search? (1/2)

- For large number, $O(\log_2 n)$ requires significantly less time than $O(n)$
- For small numbers such as $n < 25$, does not show big difference.

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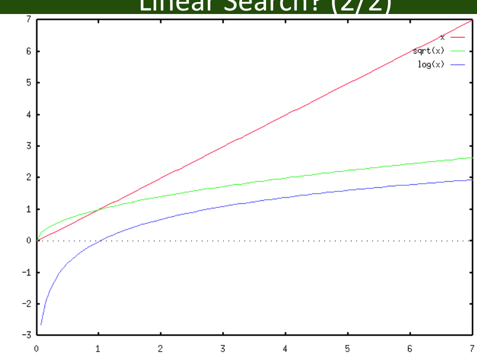
Is Binary Search is more Efficient than Linear Search? (2/2)



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Outline

- Complexity of Algorithms
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- **Sorting Algorithms and Their Efficiencies**

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Sorting Algorithm

- Organize a collection of data into either ***ascending*** or ***descending*** order.
- *Internal sort*
  - Collection of data fits entirely in the computer's main memory
- *External sort*
  - Collection of data will not fit in the computer's main memory all at once.
- We will only discuss **internal sort**.

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Selection Sort (1/2)

29	10	14	37	13	Initial Array
29	10	14	13	37	After 1st swap
13	10	14	29	37	After 2nd swap
13	10	14	29	37	After 3rd swap
10	13	14	29	37	After 4th swap

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Selection Sort (2/2)

- Comparison
$(n-1) + (n-2) + (n-3) + \ldots + 1 = n(n-1)/2$
- Exchange
$3 * (n-1)$
- Comparison + Exchange
$n(n-1)/2 + 3 * (n-1) = n^2/2 + 5n/2 - 3$
- Therefore, complexity is $O(n^2)$

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Bubble Sort (1/6)

- Compares adjacent items and exchanges them if they are out of order.
- It requires several passes over the data.
- After the first pass, the biggest item has "bubbled" to the end of the array.

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Bubble Sort (2/6)

Pass 1

29	10	14	37	13	Initial Array
10	29	14	37	13	
10	14	29	37	13	
10	14	29	37	13	
10	14	29	13	37	

Performs comparisons 4 times

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Bubble Sort (3/6)

Pass 2

10	14	29	13	37
10	14	29	13	37
10	14	29	13	37
10	14	13	29	37

Performs comparisons 3 times

Maximum number of passes?

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Bubble Sort (4/6)

- Analysis
  - At most $n - 1$ passes
  - Pass 1 requires $n - 1$ comparisons and maximum $n - 1$ exchanges
  - Pass 2 requires $n - 2$ comparisons and maximum $n - 2$ exchanges
  - .
  - .
  - Pass i requires $n - i$ comparisons and maximum $n - i$ exchanges
  - ***Until there is no exchange happens in the pass (sorted)***

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Bubble Sort (5/6)

Worst case

- Comparison:
  $(n - 1) + (n - 2) + \ldots + 1 = n\,(n - 1)/2$
- Exchanges:
  $(n - 1) + (n - 2) + \ldots + 1 = n\,(n - 1)/2$ times.
  Each exchange requires three data moves.
  Therefore, $3n(n - 1) / 2$
- Total
  $4n(n - 1)/1 = 2n(n-1) = 2n^2-2n$
  $O(n^2)$,in the worst case

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Bubble Sort (6/6)

Best case (original data is already sorted)

- Comparison: $n - 1$
- Exchanges: no exchange
- Total

  $O(n)$, in the best case

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Insertion Sort (1/6)

- Partitions the array into two regions: *sorted* and *unsorted*.
- Get the first unsorted item and find where to insert to make sorted array.
- Increase sorted part.

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Insertion sort (2/6)

If there is only ONE item, it is sorted.

Initial Array 29 | 10 | 14 | 37 | 13 (sorted | unsorted)

29 | 10 | 14 | 37 | 13 Copy 10

29 | 29 | 14 | 37 | 13 Shift 29

10 | 29 | 14 | 37 | 13 Insert 10; copy 14 (sorted | unsorted)

10 | 29 | 29 | 37 | 13 Shift 29

10 | 14 | 29 | 37 | 13 Insert 14; copy 37 (sorted | unsorted)

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Insertion sort (3/6)

10 | 14 | 29 | 37 | 13 insert 37 on top of itself (sorted | unsorted)

10 | 14 | 29 | 37 | 13 Copy 13

10 | 14 | 14 | 29 | 37 Shift 37, 29, 14

Sorted Array 10 | 13 | 14 | 29 | 37 Insert13 (sorted | unsorted)

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Insertion sort (4/6)

```
1:public static <T extends Comparable>
2:  void insertionSort (T[i] theArray, int n){
3:  for (int unsorted = 1; unsorted < n; ++unsorted){
4:      T nextItem = theArray[unsorted];
5:      int loc = unsorted;
6:      while ((loc >0) && (theArray[loc-1].compareTo(nextItem)
   >0)){
7:         theArray[loc] = theArry[loc-1];
8:         loc--;
9:      }
10:        theArry[loc] = nextItem;
11:  }
12:}
```

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Insertion sort (5/6)

- Analysis
  - (Line 3 in the code) Unsorted/sorted point will move $(n-1)$ times
    for each of the points
    - First unsorted data (m^{th}) is copied (1 move)
    - $(n\text{-}1)$
  - (Line 6 in the code) For $(m\text{ -}1)$ sorted items, check the sorted part if any item is bigger than copied item (m -1 times) and if there is sorted item larger than the copied m^{th} item, shift sorted item to the next position in the array (maximum m -1 times).
    - Insert copied m^{th} item.(1 move)
    - m grows from 1 to n-1.
    - $(1+2+\ldots+(n-1))+(1+2+\ldots+(n-1))+(n\text{-}1)$

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Insertion sort (6/6)

Total,

$(n\text{-}1)+(1+2+\ldots+(n-1))+(1+2+\ldots+(n-1))+(n\text{-}1)$

$= 2n(n\text{-}1)/2 \ + 2(n\text{-}1)$

$= \ n^2 + n - 2$

- $O(n^2)$ in the worst case

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Mergesort (1/12)

- ***Recursive sorting algorithm***
- Gives the **same** performance
- ***Divide-and-conquer***
  - Divide the array into halves
  - Sort each half
  - Merge the sorted halves into one sorted array

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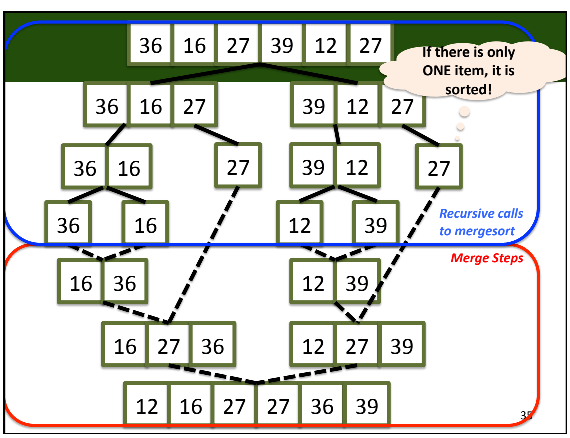

Mergesort (3/12)

```
public static
  void mergesort(Comparable[] theArray, Comparable[]
  tempArray, int first, int last){
   if (first < last){
     int mid = (first + last)/2;
     mergesort(theArray, tempArray, first, mid);
     mergesort(theArray, tempArray, mid+1, last);
     merge (theArray, tempArray, first, mid, last);
   }
}
```

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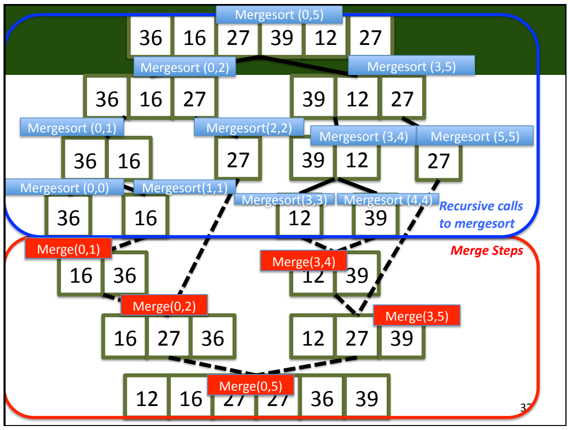

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Mergesort (5/12)

```
private static void merge (Comparable[] theArray, Comparable[]
   tempArray, int first, int mid, int last({

  int first 1 = first;
  int last 1 = mid;
  int first2 = mid+1;
  int last2 = last;
  int index = first1;

  while ((first1 <= last1) && (first2 = last2)){
    if( theArray[first1].compareTo(theArray[first2])<0) {
      tempArray[index] = theArray[first1];
      first1++;
    }
    else{
      tempArray[index] = theArray[first2];
      first2++;
    }
    index++;
  }
```

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Mergesort (6/12)

```
  while (first1 <=last1){
    tempArray[index] = theArray[first];
    first1++;
    index++;
  }
  while(first1 <= last2)
    tempArray[index] = theArray[first2];
    first2++;
    index++;
  }
  for (index = first; index <= last: ++index){
    theArray[index ] = tempArray[index];
  }
} //end merge
```

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Mergesort (7/12)

- Analysis
  - Merging:
    - for total of n items in the two array segments, at most n -1 comparisons are required.
    - n moves from original array to the temporary array.
    - n moves from temporary array to the original array.
    - Each merge step requires $3n - 1$ major operations

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Mergesort (8/12)

- Each call to `mergesort` recursively calls itself **twice**.
- Each call to `mergesort` **divides** the array into two.
  - First time: divide the array into 2 pieces
  - Second time: divide the array into 4 pieces
  - Third time: divide the array into 8 pieces

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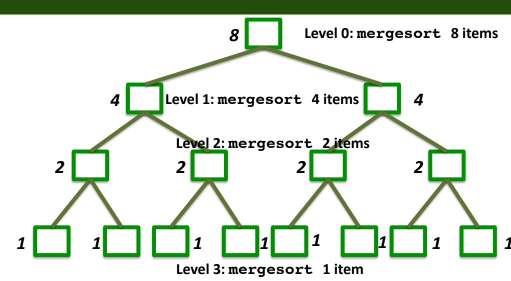
Mergesort (9/12)



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Mergesort (10/12)

- If n is a power of 2 (i.e. $n = 2^k$), then the recursion goes $k = \log_2 n$ levels deep.
- If n is not a power of 2, there are $1 + \log_2 n$ (rounded down) levels of recursive calls to `mergesort`.

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Mergesort (10/12)

- At level 0, the original call to `mergesort` calls merge once. (requires $3n - 1$ operations)
- At level 1, two calls to `mergesort` and each of them will call `merge`.
  - Total $2 * (3 * (n/2) - 1)$ operations required
- At level m, 2^m calls to `merge` occur.
  - Each of them will call merge with $n/2^m$ items and each of them requires $3(n/2^m)$-1operations. Together, $3n$-2^moperations are required.
- Because there are $log_2 n$ or 1+$log_2 n$ levels, total $O(n*log_2 n)$

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Mergesort (11/12)

- Since there are either $\log_2 n$ or $1{+}\log_2 n$ levels, `mergesort` is $O(n*\log_2 n)$ in both the **worst** and **average** cases.
- **Significantly faster** than $O(n^2)$

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Quicksort (1/9)

1. Select a **pivot** item.
2. Subdivide array into 3 parts
   - **Pivot in its sorted position**
   - Subarray with **elements < pivot**
   - Subarray with **elements >= pivot**
3. **Recursively** apply to each sub-array

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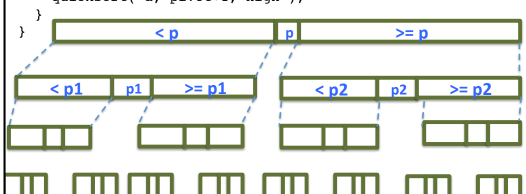
Quicksort

```
quicksort( void *a, int low, int high )  {
  int pivot;
  /* Termination condition! */
  if ( high > low )    {
     pivot = partition( a, low, high );
     quicksort( a, low, pivot-1 );
     quicksort( a, pivot+1, high );
   }
 }
```



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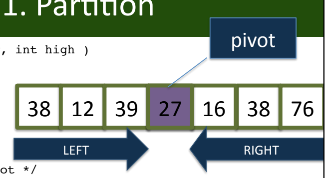
Step 1. Partition

```
int partition( void *a, int low, int high )
{
 int left, right;
 void *pivot_item;
 pivot = low + high/2;
 pivot_item = a[pivot];
 right = high; left = low;
 while ( left < right ) {
  /* Move left while item < pivot */
  while( a[left] <= pivot_item ) left++;
  /* Move right while item > pivot */
  while( a[right] > pivot_item ) right--;
  if ( left < right ) SWAP(a,left,right);
 }
 /* right is final position for the pivot */
 a[low] = a[right];
 a[right] = pivot_item;
 return right;
}
```



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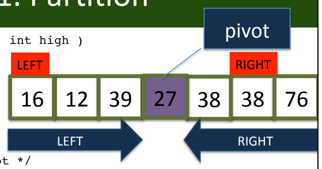
Step 1. Partition

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int partition( void *a, int low, int high )
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  while( a[left] <= pivot_item ) left++;
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 /* right is final position for the pivot */
 a[low] = a[right];
 a[right] = pivot_item;
 return right;
}
```

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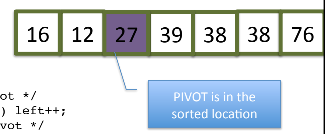
Step 1. Partition

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  while( a[left] <= pivot_item ) left++;
  /* Move right while item > pivot */
  while( a[right] > pivot_item ) right--;
  if ( left < right ) SWAP(a,left,right);
 }
 /* right is final position for the pivot */
 a[low] = a[right];
 a[right] = pivot_item;
 return right;
}
```

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Average Case

- Each level involves,
  - Maximum $(n-1)$ comparisons.
  - Maximum $(n-1)$ swaps. ($3(n-1)$ data movements)
  - $log_2\, n$ levels are required.
- Average complexity $O(n\ log_2\ n)$

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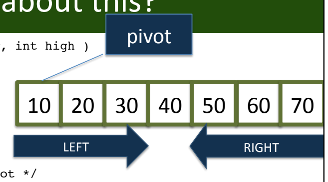
How about this?




```
int partition( void *a, int low, int high )
{
 int left, right;
 void *pivot_item;
 pivot = low;
 pivot_item = a[pivot];
 right = high; left = low;
 while ( left < right ) {
  /* Move left while item < pivot */
  while( a[left] <= pivot_item ) left++;
  /* Move right while item > pivot */
  while( a[right] > pivot_item ) right--;
  if ( left < right ) SWAP(a,left,right);
 }
 /* right is final position for the pivot */
 a[low] = a[right];
 a[right] = pivot_item;
 return right;
}
```

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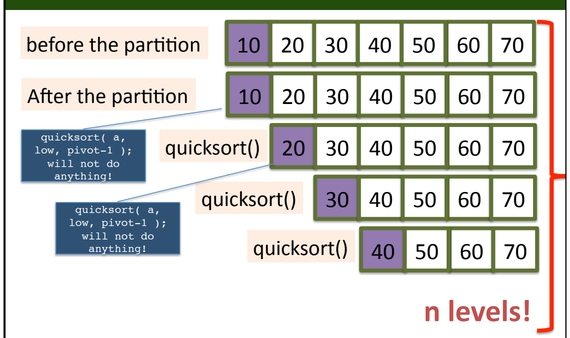
Worst Case!



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Worst case analysis

- This case involves $(n\text{-}1)+(n\text{-}2)+(n\text{-}3)+\ldots+1+0 = n(n\text{-}1)/2$ comparisons
- `Quicksort` is $O(n^2)$ for the **worst-case**.

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Selecting pivot

- Strategies for Selecting **pivot**
  - First value : worst case if the array is **sorted**.
- Middle value
  - Better for the sorted data (less exchange)
- Median of 3 sample values
  - Worst case can still happen but less likely

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Radix Sort

1. Take the least **significant** digit
2. **Sort** the numbers based on the digit
3. **Concatenate** the buckets together in order
4. **Recursively** sort each bucket, starting with the next digit to the right

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Example

0123, 2154, 0222, 0004, 0283, 1560, 1061, 2150

(156**0**, 125**0**) (106**1**) (022**2**) (012**3**, 028**3**) (215**4**, 000**4**)
1560, 1250,1061,0222,0123, 0283,2154, 0004

(00**0**4)(02**2**2,01**2**3)(21**5**0,21**5**4)(15**6**0,10**6**1)(02**8**3)
0004, 0222,0123,2150,2154,1560,1061,0283

(0**0**04,1**0**61)(0**1**23,2**1**50,2**1**54)(0**2**22, 0**2**83)(1**5**60)
0004,1061,0123,2150,2154,0222, 0283,1560

(**0**004,**0**123, **0**222, **0**283)(**1**061, **1**560)(**2**150, **2**154)
0004, 0123, 0222, 0283, 1061, 1560, 2150, 2154

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A radix sort of n decimal integers of d digits each

```
radixSort(inout theArry:ItemArray, in n:integer, in
  d:integer){
  for (j=d down to 1)
    initialize 10 groups to empty
    initialize a counter for each group to 0
    for (i = 0 through n-1) {
      k = jth digit of theArray[i]
      place theArray[i] at the end of group k
      increase kth counter by 1
      }
    replace the items in theArray with all the items in
     group 0, followed by all the items in group1, and so
     on.
 }
```

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Radix sort

- Analysis
  - n moves each time it forms groups
  - n moves to combine them again into one group.
  - Total $2n*d$ (for the strings of d characters)
  - Radix sort is $O(n)$ for $d << n$

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Next reading

- Rosen Ch 8.1 (for 6th edition: 7.1) Recurrence Relations
- Rosen Ch 8.3 (for 6th edition: 7.3) Divide and Conquer

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