Part 9. Relations

CS 200 Algorithms and Data Structures

1

Outline

- **Relations and Their Properties**
- n-ary Relations and Their Applications
- Representing Relations

2

Relations on a Set

- A relation on a set A defines a relationship between elements of A
- Example: relations on the set of integers

$R_1 = \{(a,b) \mid a \leq b\}$

$R_2 = \{(a,b) \mid a > b\}$

$R_3 = \{(a,b) \mid a = b + 1\}$

Relations on a Set as Graphs

- Consider the relation R on cities:

$R = \{(a,b) \mid a, b$ are cities such that the population of a is smaller than that of $b\}$

- We can represent R as a **directed graph** where there is an edge from a to b if (a,b) is in R.

Binary Relations

Example 1

A – set of students

B – set of courses

R – pairs (a,b) such that student a is enrolled in course b

R = {(chris, cs200), (mike, cs520),…}

Example 2

A – set of cities B – set of US states

R – (a,b) such that city a is in state b

R = {(Denver, CO), (Laramie, WY),…}

Binary Relations

- A binary relation from a set A to a set B is a set R of ordered pairs (a,b) where $a \in A$ and $b \in B$.
- The notation aRb denotes $(a,b) \in R$
- Example: $A = \{0,1,2\}$, $B = \{a,b\}$ and $R=\{(0,a),(0,b),(1,a),(2,b)\}$

Relations and Cartesian Products

- Let A,B be sets
- The cartesian product of A and B is denoted by $A \times B$ and is equal to:
  $\{ (a,b) \mid a \in A \text{ and } b \in B\}$
- A binary relation from A to B is a subset of $A \times B$
- Given sets A and B with sizes n and m respectively
  - The number of elements in $A \times B$ is $n \times m$ and
  - The number of binary relations from A to B is 2^{nm}

Reflexive relation

- A relation R on a set A is called ***reflexive*** if $(a,a) \in R$ for every element $a \in A$.

8

Example: Which of these relations are *reflexive*?

- Relations on {1,2,3,4}

R_1 = {(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)}
R_2 = {(1,1),(1,2),(2,1)}
R_3 = {(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)}
R_4 = {(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)}
R_5 = {(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)}
R_6 = {(3,4)}

Answer: R_3 and R_5

9

Example

- On a set S of n elements, how many of reflexive relations are there?

10

Symmetric relation

- A relation R on a set A is called **symmetric** if $(b,a) \in R$ whenever $(a,b) \in R$, for all $a, b \in A$.
- A relation R on a set A such that for all $a,b \in A$, if $(a,b) \in R$ and $(b,a) \in R$, then $a = b$ is called **antisymmetric**

11

Which of these relations are **symmetric**?

- Relations on {1,2,3,4}

R_1 ={(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)}
R_2 = {(1,1),(1,2),(2,1)}
R_3 = {(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)}
R_4 = {(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)}
R_5 = {(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)}
R_6 = {(3,4)}

Answer: R_2 and R_3

12

Example

- On a set S of n elements, how many of symmetric relations are there?

13

Which of these relations are **antisymmetric**?

- Relations on {1,2,3,4}

R_1 ={(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)}
R_2 = {(1,1),(1,2),(2,1)}
R_3 = {(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)}
R_4 = {(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)}
R_5 = {(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)}
R_6 = {(3,4)}

Answer: R_4 R_5 and R_6

14

Transitive Relation

- A relation R on a set A is called **transitive** if whenever (a,b) $\in R$ and (b,c) $\in R$, then (a,c) $\in R$ for all a,b,c $\in A$

15

Which of the relations are transitive?

- Relations on {1,2,3,4}

R_1 ={(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)}
R_2 = {(1,1),(1,2),(2,1)}
R_3 = {(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)}
R_4 = {(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)}
R_5 = {(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)}
R_6 = {(3,4)}

Answer: R_4 R_5 and R_6

16

Combining Relations

Let A = {1,2,3} and B = {1,2,3,4}.

The relations $R_1 = \{(1,1),(2,2),(3,3)\}$ and $R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$ can be combined to obtain:

$R_1 \cup R_2$ =

$R_1 \cap R_2$ =

$R_1 - R_2$ =

$R_2 - R_1$ =

17

Combining Relations

Let A = {1,2,3} and B = {1,2,3,4}.

The relations $R_1 = \{(1,1),(2,2),(3,3)\}$ and $R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$ can be combined to obtain:

$R_1 \cup R_2$ = {(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)}

$R_1 \cap R_2$ = {(1,1)}

$R_1 - R_2$ = {(2,2),(3,3)}

$R_2 - R_1$ = {(1,2),(1,3),(1,4)}

18

Composite of relations

- Let R be a relation from a set A to a set B and S be a relation from B to a set C.
- The **composite** of R and S is the relation consisting of ordered pairs (a,c), where a $\in A$, c $\in C$, and for which there exists an element b $\in$ B such that (a,b) $\in R$ and (b,c) $\in$ S .
- We denote the composite of R and S by $S \circ R$.

19

Example

- What is the composite of the relation R and S, where R is the relation from {1,2,3} to {1,2,3,4} with R = {(1,1),(1,4),(2,3),(3,1),(3,4)} and S is the relation from {1,2,3,4} to {0,1,2} with S = {(1,0),(2,0),(3,1),(3,2),(4,1)}?

- Solution:

  $S \circ R$ = {(1,0),(1,1),(2,1),(2,2),(3,0),(3,1)}

20

The Powers of a relation R

- Let R be a relation on the set A. The powers R^n, n = 1,2,3,…, are defined recursively by

  $R^1 = R$ and $R^{n+1} = R^n \circ R$

21

Example

- Let R = {(1,1),(2,1),(3,2),(4,3)}. Find the powers R^n, n = 2,3,4…

Solution:

$R^2 = R \circ R$, R^2 = {(1,1),(2,1),(3,1)(4,2)}

$R^3 = R^2 \circ R$, R^3= {(1,1), (2,1), (3,1), (4,1)}

22

Theorem 9-1

- The relation R on a set A is transitive if and only if $R^n \subseteq R$ for n = 1,2,3,…

Proof

23

Outline

- Relations and Their Properties
- ***n*-ary Relations and Their Applications**
- Representing Relations

24

n-ary Relations

Definition: Let $A_1, A_2, \ldots, A_n$ be sets. An *n*-ary relation on these sets is a subset of $A_1 \times A_2 \times \ldots \times A_n$.

The sets $A_1, A_2, \ldots, A_n$ are called the *domains* of the relation, and n is called its *degree*.

Example: The *between* relation consisting of triples (a,b,c) where a,b,c are integers such that a < b < c

Example

Relation R:
Person X supports that Person Y helps Person Z

Person X	Person Y	Person Z
Alice	Bob	Denise
Charles	Alice	Bob
Charles	Charles	Charles
Denise	Denise	Denise

n-Tuples

- An ordered n-tuple is a sequence of n objects
  $(x_1, x_2, \ldots, x_n)$
  First component is x_1
  …
  n-th component is x_n
- An ordered pair: 2-tuple (x, y)
- An ordered triple: 3-tuple (x, y, z)

Tuples vs Sets

- Two tuples are equal iff they are equal coordinate-wise
  $(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n)$ iff
  $x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n$
- $(2, 1) \neq (1, 2)$, but $\{2, 1\} = \{1, 2\}$
- $(1, 2, 1) \neq (2, 1)$, but $\{1, 2, 1\} = \{2, 1\}$

Databases and Relations

What can be the primary keys?

Students			
StudentName	IDnumber	Major	GPA
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

Databases defined by relations are called *relational databases*

Operations on n-ary Relations

- Form new n-ary relations.
  - Determining all n-tuples in the n-ary relation that satisfy certain conditions
  - Find all students who have a grade point average above 3.5

Selection Operator

- Let R be an n-ary relation and C a condition that elements in R may satisfy.
- Then the selection operator S_c maps the n-ary relation R to the n-ary relation of all n-tuples from R that satisfy the condition C.

31

Projections

- The projection $P_{i1i2,..im}$, where $i_1 < i_2 2 < \ldots < i_m$, maps the n-tuple $(a_1,a_2,\ldots,a_n)$ to the m-tuple $(a_1,a_2,\ldots,a_m)$ where $m \leq n$.

32

Example

Students			
StudentName	IDnumber	Major	GPA
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

What relation results when the projection P1,4 is applied to this relation?

33

Example

Students	
StudentName	GPA
Ackermann	3.88
Adams	3.45
Chou	3.49
Goodfriend	3.45
Rao	3.90
Stevens	2.99

Column 2 and 3 are deleted.

34

Outline

- Relations and Their Properties
- n-ary Relations and Their Applications
- **Representing Relations**

35

Representing Relations Using Matrices

- A relation between finite sets can be represented using a zero-one matrix.
- R is a relation from $A = \{a_1, a_2,\ldots, a_m\}$ to $B = \{b_1, b_2,\ldots, b_n\}$
- Using Matrix $\mathbf{M}_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

36

Example 1

$A = \{1,2,3\}$

$B = \{1,2\}$

Relation R from A to B containing (a, b) if $a \in A, b \in B$, and $a>b$.

What is the matrix representing R if $a_1 = 1$, $a_2 = 2$ and $a_3 = 3$ and $b_1 = 1$ and $b_2 = 2$?

37

Example 2

- Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, \dots, b_5\}$. Which ordered pairs are in the relation R represented by the matrix,

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

38

Reflexive? Symmetric? Antisymmetric?

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

39

Example 3

- Find the matrix representing the relations $S \circ R$, where the matrices representing R and S are,

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{M}_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

40

Example 3 - continued

- Boolean product of $\mathbf{M}_R$ and $\mathbf{M}_S$

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{M}_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$C_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \dots \vee (a_{ik} \wedge b_{kj})$$

$$M_R \otimes M_S =$$

$$\begin{bmatrix} (M_{R0,0} \wedge M_{S0,0}) \vee (M_{R0,1} \wedge M_{S1,0}) \vee (M_{R0,2} \wedge M_{S2,0}) & (M_{R0,0} \wedge M_{S0,1}) \vee (M_{R0,1} \wedge M_{S1,1}) \vee (M_{R0,2} \wedge M_{S2,1}) & (M_{R0,0} \wedge M_{S0,2}) \vee (M_{R0,1} \wedge M_{S1,2}) \vee (M_{R0,2} \wedge M_{S2,2}) \\ (M_{R1,0} \wedge M_{S0,0}) \vee (M_{R1,1} \wedge M_{S1,0}) \vee (M_{R1,2} \wedge M_{S2,0}) & (M_{R1,0} \wedge M_{S0,1}) \vee (M_{R1,1} \wedge M_{S1,1}) \vee (M_{R1,2} \wedge M_{S2,1}) & (M_{R1,0} \wedge M_{S0,2}) \vee (M_{R1,1} \wedge M_{S1,2}) \vee (M_{R1,2} \wedge M_{S2,2}) \\ (M_{R2,0} \wedge M_{S0,0}) \vee (M_{R2,1} \wedge M_{S1,0}) \vee (M_{R2,2} \wedge M_{S2,0}) & (M_{R2,0} \wedge M_{S0,1}) \vee (M_{R2,1} \wedge M_{S1,1}) \vee (M_{R2,2} \wedge M_{S2,1}) & (M_{R2,0} \wedge M_{S0,2}) \vee (M_{R2,1} \wedge M_{S1,2}) \vee (M_{R2,2} \wedge M_{S2,2}) \end{bmatrix}$$

41

Representing Relations Using Digraphs

- Each element of the set is represented by a point, and each ordered pictorial represented using an arc with its direction indicated by an arrow.

42

Directed Graph

- A directed graph, or digraph, consists of a set V of ***vertices*** (or nodes) together with a set E of ordered pairs of elements of V called ***edges*** (or arcs).
  - The vertex a is called the ***initial vertex*** of the edge (a,b) and the vertex b is called the ***terminal vertex*** of this edge.

43

Example 1

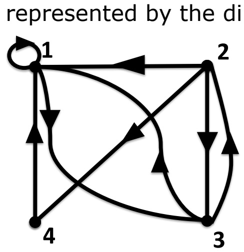
The directed graph of the relation

R = {(1,1),(1,3),(2,1),(2,3),(2,4),(3,1), (3,2),(4,1)} on the set {1,2,3,4}

44

Example 2

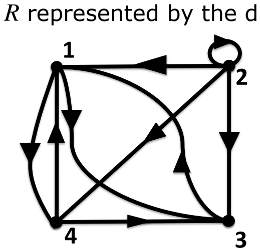
- What are the ordered pairs in the relation R represented by the directed graph?



45

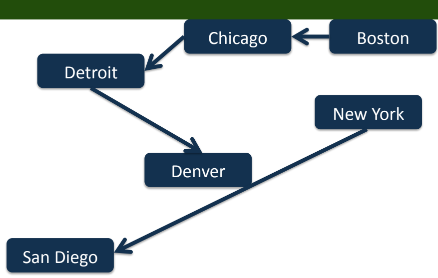
Example 3

- What are the ordered pairs in the relation R represented by the directed graph?



46

Closures of Relations



- R is a relation containing (a,b) if there is a direct data transfer facility.
- Can we send a data from x to y?

47

Transitive closure of R

- The smallest transitive relation that contains R.

48

Reflexive closure of R

- R = {(1,1),(1,2),(2,1),(3,2)} on the set A={1,2,3}
- The smallest reflexive relation containing R?
  - Adding (2,2) and (3,3) to R
- The reflexive closure of R
  - {(1,1),(1,2),(2,1),(3,2),(2,2),(3,3)}
- Add a diagonal relation

$R \cup \Delta, where \Delta = \{(a,a) \mid a \in A\}$

49

Example

- What is the reflexive closure of the relation R={(a,b)|a<b}?

50

Symmetric closure of R

- {(1,1),(1,2),(2,2),(2,3),(3,1),(3,2)} on {1,2,3}
- How can we produce a symmetric relation that is as small as possible and contains R?
- Add (2,1) and (3,1)
- Symmetric closure of R
  - {(1,1),(1,2),(2,2),(2,3),(3,1),(3,2),(2,1), (3,1)}

51

Constructing the symmetric closure of R

- Add all ordered pairs of the form (b,a), where (a,b) is in the relation, that are not already present in R.
- $R \cup R^{-1}$ is the symmetric closure of R where, $R^{-1} = \{(b,a) \mid (a,b) \in R\}$

52

Example

- What is the symmetric closure of the relation R={(a,b)|a>b} on the set of positive integer?

53

Paths in Directed Graphs

- Definition
  - A path from a to b in the directed graph G is a sequence of edges $(x_0, x_1), (x_1, x_2)\ldots(x_{n-1}, x_n)$ in G, where *n* is a nonnegative integer, and $x_0 = a$ and $x_n = b$, that is, a sequence of edges where the terminal vertex of an edge is the same as the initial vertex in the next edge in the path.

54

Paths

- Path
  - $x_0, x_1, x_2, \ldots, x_n$ and has length n.

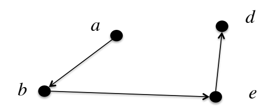
– The empty set of edges
  - A path from a to a

– A circuit of cycle
  - A path of length n>=1 that begins and ends at the same vertex

55

Example

- What is the length of path, a,b,e,d?



56

Connectivity relation

- The connectivity relation R^* consists of the pairs (a,b) such that there is a path of length at least one from a to b in R

57

Example

- Let R be the relation on the set of all people in the world that contains (a,b) if a has met b. What is R^n, where n is a positive integer greater than one? What is R^*?

58

Equivalence Relations

- Definition
  – A relation on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.

59

Example

- Let R be the relation on the set of integers such that aRb if and only if $a=b$ or $a=-b$. Is this relation equivalence relation?

60