Bipartite Analogues of Comparability and Cocomparability Graphs

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Abstract

We propose bipartite analogues of comparability and cocomparability graphs. Surprisingly, the two classes coincide. We call these bipartite graphs cocomparability bigraphs. We characterize cocomparability bigraphs in terms of vertex orderings, forbidden substructures, and orientations of their complements. In particular, we prove that cocomparability bigraphs are precisely those bipartite graphs that do not have edge-asteroids; this is analogous to Gallai's structural characterization of cocomparability graphs by the absence of (vertex-) asteroids. Our characterizations imply a robust polynomial-time recognition algorithm for the class of cocomparability bigraphs. Finally, we also discuss a natural relation of cocomparability bigraphs to interval containment bigraphs, resembling a well-known relation of cocomparability graphs to interval graphs.

Key words: Cocomparability bigraph, chordal bigraph, interval bigraph, interval containment bigraph, two-directional orthogonal-ray graph, characterization, orientation, vertex ordering, invertible pair, asteroid, edge-asteroid, recognition, polynomial time algorithm.

1 Introduction

In this paper we propose bipartite analogues of two popular graph classes, namely, the comparability and the cocomparability graphs [11]. Interestingly, the two analogues coincide, and we obtain just one new class of bigraphs. This class exhibits some features of both comparability and cocomparability graphs, but the similarities are significantly

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stronger with the class of cocomparability graphs; therefore in this paper we call these bigraphs *cocomparability bigraphs*.

Cocomparability graphs are usually defined as the complements of comparability (i.e., transitively orientable) graphs, and their name reflects this fact. However, they are a natural and interesting graph class on their own, can be defined independently of their complements, and have an elegant forbidden substructure characterization [11]. Our cocomparability bigraphs bear strong resemblance to these properties.

We remind the reader that many popular graph classes have bipartite analogues. For instance, for chordal graphs there is a well-known bipartite analogue, namely, chordal bipartite graphs, or *chordal bigraphs*. They have a similar ordering characterization, forbidden substructure characterization, and even a geometric representation characterization [12, 16].

Interval graph analogues have a more complex history: the bipartite analogues studied first, namely interval bigraphs [13, 26, 28], do not share many nice properties of interval graphs – in particular there is no known forbidden substructure characterization. A better bipartite analogue of interval graphs turned out to be the interval containment bigraphs discussed below. This class has many similar properties and characterizations to the class of interval graphs, in particular an ordering characterization, and a forbidden substructure characterization [8].

When considering what constitutes a natural bigraph analogue of a graph class, it turns out best to be guided by the ordering characterizations. Especially revealing are the (equivalent) matrix formulations of the ordering characterizations. Take the case of chordal graphs. A graph is *chordal* if it does not contain an induced cycle of length greater than three. Chordal graphs are characterized by the existence of a perfect elimination ordering. An ordering $\prec$ of the vertices of a graph G is a *perfect elimination ordering* if $u \prec v \prec w$ and $uv \in E(G), uw \in E(G)$ implies that $vw \in E(G)$. To consider the matrix formulation, it is most convenient to think of graphs as *reflexive*, i.e., having a loop at each vertex. (This makes sense, for instance, for the geometric characterization of chordal graphs as intersection graphs of subtrees of a tree: since each of the subtrees intersects itself, each vertex has a loop.) The matrix condition is expressed in terms of the adjacency matrix of G; because of the loops, the matrix has all 1's on the main diagonal. The Γ *matrix* is the two-by-two matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \tag{1}$$

The Γ matrix is a *principal submatrix* of an adjacency matrix of a reflexive graph if any of the entries 1 lies on the main diagonal. Then a perfect elimination ordering of the vertices of G corresponds to a simultaneous permutation of rows and columns of the adjacency matrix so the resulting form of the matrix has no Γ as a principal submatrix. In other words, *a reflexive graph G is chordal if and only if its adjacency matrix can be permuted, by simultaneous row and column permutations, to a form that does not have the Γ matrix as a principal submatrix*. For bipartite graphs, we use the *bi-adjacency matrix*, in which rows correspond to vertices of one part and columns to vertices of the other part, with an entry 1 in a row and a column if and only if the two corresponding vertices are adjacent. Note that this means that re-ordering of the vertices corresponds to independent permutations of rows and columns. Chordal bigraphs have an ordering characterization [2] which translates to the following matrix formulation. *A bipartite graph G is a chordal bigraph if and only if its bi-adjacency matrix can be permuted, by independent row and column permutations, to a form that does not have the Γ matrix as a submatrix.* This indeed yields a class with nice properties analogous to chordal graphs. In particular, a bipartite graph is a chordal bigraph if and only if it does not contain an induced cycle of length greater than four [12].

We next look at the case of interval graphs. A graph is an *interval graph* if it is the intersection graph of a family of intervals in the real line. As for chordality, it is most natural to consider these to be reflexive graphs. Interval graphs are characterized by the existence of an ordering $\prec$ of $V(G)$ such that if $u \prec v \prec w$ and $uw \in E(G)$, then $vw \in E(G)$. The *Slash matrix* is the two-by-two matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2}$$

The *Slash* matrix is a *principal submatrix* of an adjacency matrix of a reflexive graph if either entry 1 lies on the main diagonal. We may now reformulate the ordering characterization as the following matrix characterization. *A reflexive graph G is an interval graph if and only if its adjacency matrix can be permuted, by simultaneous row and column permutations, to a form that does not have the Γ or the* Slash *matrix as a principal submatrix.*

A bipartite graph G, with bipartition (X, Y), is an *interval containment bigraph* if there is a family of intervals I_v, $v \in X \cup Y$, such that for $x \in X$ and $y \in Y$, we have $xy \in E(G)$ if and only if I_x contains I_y. The intervals I_v, $v \in X \cup Y$, appear to depend on the bipartition (X, Y) in the definition, but it is easy to see that if G is an interval containment bigraph with respect to one bipartition (X, Y), it remains so with respect to any other bipartition [16]. Simple transformations show that interval containment bigraphs coincide with two other previously investigated classes of bipartite graphs [16], namely, two-directional orthogonal-ray graphs [30], and complements of circular arc graphs of clique covering number two [14]. These graphs can be characterized by the existence of orderings $\prec_X, \prec_Y$ such that for $u, v \in X$ and $w, z \in Y$, if $u \prec_X v, w \prec_Y z$ and $uz, vw \in E(G)$ then $vz \in E(G)$. This implies that *a bipartite graph G is an interval containment bigraph if and only if its bi-adjacency matrix can be permuted, by independent row and column permutations, to a form that does not have the Γ or the* Slash *matrix as a submatrix.*

Interval graphs have elegant structural characterizations. An *asteroid* in a graph is a set of $2k+1$ vertices $v_0, v_1, \ldots, v_{2k}$ (with $k \geq 1$) such that for each $i = 0, 1, \ldots, 2k$, there is a path joining v_{i+k} and v_{i+k+1} whose vertices are not neighbours of v_i (subscript additions are modulo $2k + 1$). An asteroid with three vertices ($k = 1$) is called an *asteroidal triple*. Lekkerkerker and Boland [21] proved that a graph is an interval graph if and only if it has no induced cycle of length greater than three and no asteroidal triple. Gilmore and Hoffman [10] showed that a graph is an interval graph if and only if it is chordal and cocomparability. According to Gallai [9], cocomparability graphs are precisely the graphs that do not contain asteroids. Therefore *a graph is an interval graph if and only if it contains no induced cycle of length greater than three and no asteroid.*

The interval containment bigraphs defined above have an analogous structural characterization. As is often the case with bigraph analogues, we must first translate vertex properties into edge properties. An *edge-asteroid* in a bipartite graph consists of a set of $2k+1$ edges $e_0, e_1, \ldots, e_{2k}$ (with $k \geq 1$) such that for each $i = 0, 1, \ldots, 2k$, there is a walk joining e_{i+k} and e_{i+k+1} (including both end vertices of e_{i+k} and e_{i+k+1}) that contains no vertex adjacent to either end of e_i (subscript additions are modulo $2k+1$). It follows from [8] that *a bipartite graph G is an interval containment bigraph if and only if it contains no induced cycle of length greater than four and no edge-asteroid.* Figure 1 depicts an edge-asteroid and a bipartite graph that contains an edge-asteroid with five edges but neither an edge-asteroid with three edges nor an induced cycle of length greater than four. We note that in the example there are paths joining the consecutive edges; while this can be ensured in general, we choose to work with walks for technical reasons.

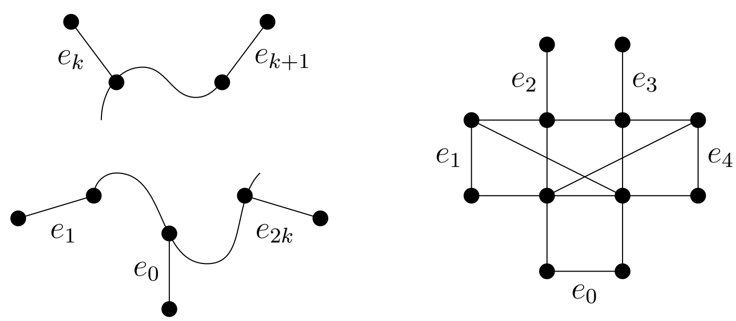

Figure 1: Edge-asteroid

Armed with these examples we now explore the bipartite analogues of comparability and cocomparability graphs. It turns out it is most natural to take cocomparability graphs as reflexive, and comparability graphs as irreflexive (i.e., without loops). (A hint to the former may be the above-mentioned theorem of Gilmore and Hoffman. Since both interval and chordal graphs are reflexive, it makes sense to insist that cocomparability graphs also be reflexive.) The I_2 *matrix* is the two-by-two identity matrix, i.e.,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3}$$

Note that the I_2 matrix is obtained from the *Slash* matrix by the exchange of the two rows, a row permutation. The I_2 matrix is a *principal submatrix* of an adjacency matrix of an irreflexive graph if either of the entries 0 lies on the main diagonal. It is well known that a graph G is a comparability graph if and only if it has an ordering $\prec$ of $V(G)$ such that $u \prec v \prec w$ and $uv \in E(G), vw \in E(G)$ implies that $uw \in E(G)$ [6]. This has a natural matrix formulation as follows. *An irreflexive graph is a comparability graph if and only if its adjacency matrix can be permuted, by simultaneous row and column permutations, to a form that does not have the I_2 matrix as a principal submatrix.* Correspondingly, we define a bipartite graph G to be a *comparability bigraph* if its bi-adjacency matrix can be permuted, by independent row and column permutations, to a form that does not have the I_2 matrix as a submatrix.

The situation is similar for cocomparability graphs. Obviously, a graph is a cocomparability graph if and only if it has an ordering $\prec$ of $V(G)$ (called a *cocomparability ordering*) such that $u \prec v \prec w$ and $uv \notin E(G)$, $vw \notin E(G)$ implies that $uw \notin E(G)$. In matrix terms, *a reflexive graph is a cocomparability graph if and only if its adjacency matrix can be permuted, by simultaneous row and column permutations, to a form that does not have the* Slash *matrix as a principal submatrix.* Thus we define a bipartite graph G to be a *cocomparability bigraph* if its bi-adjacency matrix can be permuted, by independent row and column permutations, to a form that does not have the *Slash* matrix as a submatrix. By the previous observation about the relation of the I_2 and the *Slash* matrices, we see that by reversing the rows of a bi-adjacency matrix without the I_2 matrix, we obtain a matrix without the *Slash* matrix. Hence we can conclude the following fact mentioned earlier.

Proposition 1.1. *A bipartite graph is a comparability bigraph if and only if it is a cocomparability bigraph.* □

Consequently, we shall call our class just by one name. We choose to call these bipartite graphs cocomparability bigraphs, as they exhibit more similarities with cocomparability graphs.

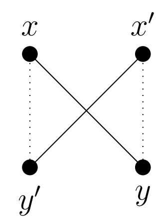

Figure 2: The forbidden subgraph S in G.
(The top and bottom vertices are ordered left-to-right according to $\prec_X, \prec_Y$ respectively.)

For future reference, we reformulate the matrix definition in terms of vertex ordering. The *Slash* matrix corresponds to the pattern S in Figure 2, in the following sense: in the figure we have a bipartition of a bipartite graph into vertices in the upper row, coming from a set X, and those in the lower row, coming from a set Y. The set X will correspond to the rows of the matrix and the set Y to the columns of the matrix. The independent orderings of the rows and columns yield two orderings, the ordering $\prec_X$ of the set X, and the ordering $\prec_Y$ of the set Y. The depicted pattern S describes precisely the presence of a *Slash* submatrix. Therefore, for a bipartite graph G, with bipartition (X, Y), we say that the pair of orderings $\prec_X$, $\prec_Y$ is *S-free*, if for all $u, v \in X$ and $w, z \in Y$ with

$u \prec_X v, w \prec_Y z$, $uz, vw \in E(G)$ imply $uw \in E(G)$ or $vz \in E(G)$. Therefore, we have the following:

Proposition 1.2. *A bipartite graph, with bipartition (X, Y), is a cocomparability bigraph if and only if there exists an S-free pair of orderings $\prec_X, \prec_Y$ of X, Y respectively.* □

It is easy to see that a bipartite graph with a bipartition (X, Y) has an S-free pair of orderings $\prec_X, \prec_Y$ of X, Y respectively, if and only if this is true in every bipartition.

Proposition 1.2 is highly reminiscent of a characterization of *bipartite permutation graphs* (i.e., those bipartite graphs that are also cocomparability graphs), studied in [31]. For a bipartite graph G, with bipartition (X, Y), we say that a pair of orderings $\prec_X, \prec_Y$ is *strongly S-free* if for all $u, v \in X$ and $w, z \in Y$ with $u \prec_X v, w \prec_Y z$, $uz, vw \in E(G)$ imply $uw \in E(G)$ and $vz \in E(G)$. The following characterization of bipartite permuatation graphs is given in [31]; it shows, in particular, that bipartite permutation graphs are a subclass of cocomparability bigraphs.

Proposition 1.3. *[31] A bipartite graph, with bipartition (X, Y), is a bipartite permutation graph if and only if there exists a strongly S-free pair of orderings $\prec_X, \prec_Y$.* □

Suppose G is a bipartite graph with bipartition (X, Y). We define two edges $xy, x'y'$ with $x, x' \in X, y, y' \in Y$ to be *independent* if they are disjoint and neither xy' nor $x'y$ is an edge of G. Given a pair of orderings $\prec_X$, $\prec_Y$ of X, Y respectively, we say that two edges $xy, x'y'$ with $x, x' \in X, y, y' \in Y$ *cross* if $x \prec_X x'$ and $y' \prec_Y y$, or $x' \prec_X x$ and $y \prec_Y y'$. It is clear that the pair $\prec_X$, $\prec_Y$ is S-free if and only if no two independent edges cross.

We define the *independence graph* $I(G)$ of G, whose vertices are the edges of G, and two vertices of $I(G)$ are adjacent just if their corresponding edges are independent. Thus the complement $\overline{I(G)}$ has two vertices adjacent if and only if the corresponding edges in G share an end or are joined by at least one other edge. Therefore the edges of a walk in G correspond to the vertices of a walk in $\overline{I(G)}$, while the vertices of a walk in $\overline{I(G)}$ correspond to a set of edges in G which contains a set of edges of a walk.

Proposition 1.4. *Let G be a bipartite graph. Then G contains an edge-asteroid if and only if $\overline{I(G)}$ contains an asteroid.* □

Let us now point out some similarities of cocomparability bigraphs and cocomparability graphs.

Cocomparability graphs admit an elegant forbidden structure characterization in terms of asteroids. *A graph is a cocomparability graph if and only if it has no asteroids* [9]. Note that it can be deduced from this that *a graph is an interval graph if and only if it is both a chordal graph and a cocomparability graph* [10].

Our main theorem in this paper asserts that *a bipartite graph is a cocomparability bigraph if and only if it does not contain an edge-asteroid.* This implies that *a bipartite graph is an interval containment bigraph if and only if it is both a chordal bigraph and a* *cocomparability bigraph.* These two results strongly resemble the corresponding statements for cocomparability graphs and interval graphs discussed above, and make a good case that these are indeed the right analogues.

This paper is organized as follows. In Section 2 we translate the ordering properties of cocomparability bigraphs into properties of orientations of the complements, and we identify two forbidden structures for the existence of such orientations, namely, invertible pairs and edge-asteroids. We show that each of these structures can be used to characterize bigraphs whose complements have suitable orientations. In Section 3 we prove that if a bigraph contains no invertible pair then it is a cocomparability bigraph. It follows that each of the forbidden structures also characterizes cocomparability bigraphs. Finally, in Section 4, we show that cocomparability bigraphs are recognizable in polynomial time and point out some consequences of our characterizations of cocomparability bigraphs.

2 Orientations and obstructions

Let G be a bipartite graph with bipartition (X, Y) and let $\overline{G}$ be the complement of G. Then X and Y each induces a complete subgraph in $\overline{G}$. We shall consider mixed graphs that are obtained from $\overline{G}$ by orienting some of the edges in the complete subgraphs induced by X and Y. We call such a mixed graph a *special orientation* of $\overline{G}$. Note that in a special orientation of $\overline{G}$, no edge of $\overline{G}$ between X and Y gets oriented. A special orientation of $\overline{G}$ is *full* if all edges in the complete subgraphs induced by X and Y are oriented.

Let $\vec{G}$ be a special orientation of $\overline{G}$. Suppose that $xx'yy'$ is an induced 4-cycle in $\overline{G}$ where $x, x' \in X$ and $y, y' \in Y$ (i.e., $xy, x'y'$ are independent edges of G). We say that x, x', y, y' induce the *pattern* T in $\vec{G}$ if xx' and $y'y$ are oriented (see Figure 3). We call $\vec{G}$ *T-free* if it does not contain the pattern T and *acyclic* if it does not contain a directed cycle.

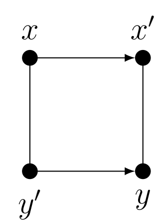

Figure 3: The forbidden pattern T in a special orientation of $\overline{G}$

Suppose that G is a cocomparability bigraph and $\prec_X$, $\prec_Y$ is an S-free pair of orderings of X, Y respectively. We construct a full special orientation of $\overline{G}$ by orienting an edge from u to v if and only if $u \prec_X v$ or $u \prec_Y v$. It is clear that this orientation is T-free; moreover, it is also *acyclic*, i.e., there is no directed cycle in X or in Y. Conversely, suppose that $\overline{G}$ has a full special orientation that is acyclic and T-free. Then X and Y admit orderings $\prec_X, \prec_Y$ respectively such that $x \prec_X x'$ if and only if xx' is oriented and

$y \prec_Y y'$ if and only if yy' is oriented. This pair of orderings $\prec_X, \prec_Y$ is S-free. Therefore we have the following:

Proposition 2.1. *Let G be a bipartite graph with bipartition (X, Y). Then the following statements are equivalent:*

(i) G is a cocomparability bigraph;

(ii) $\overline{G}$ has a full special orientation that is acyclic and T-free. □

We will first study when $\overline{G}$ has a full special orientation that is T-free (not necessarily acyclic). There are two natural obstructions for this to happen.

We say that two walks in G that both begin in X or both begin in Y are *congruent* if they have the same length, and if for each i their i-th edges are independent. A walk in G is an (a, b)-*walk* if it starts in a and ends in b. A pair of vertices u, v in G is called an *invertible pair* if there exist congruent walks W, W' where W is a (u, v)-walk and W' is a (v, u)-walk.

Proposition 2.2. *If a bipartite graph G contains an invertible pair, then $\overline{G}$ does not have a full special orientation that is T-free.*

Proof: Suppose to the contrary that $\overline{G}$ has a full special orientation $\vec{G}$ that is T-free. Let u, v be an invertible pair, with (u, v)-walk $W : u_0u_1 \ldots u_k$ and (v, u)-walk $W' : v_0v_1 \ldots v_k$ in G such that u_iu_{i+1} and v_iv_{i+1} are independent for each i. Assume without loss of generality that u_0v_0 is an oriented edge in $\vec{G}$. Since $\vec{G}$ is T-free and u_iu_{i+1} and v_iv_{i+1} are independent for each i, u_iv_i is an oriented edge in $\vec{G}$ for each i. In particular, u_0v_0 and u_kv_k are oriented edges in $\vec{G}$. But $u_0 = u = v_k$ and $v_0 = v = u_k$, and hence uv and vu are both oriented edges in $\vec{G}$, a contradiction. □

Corollary 2.3. *If a bipartite graph G contains an invertible pair, then G is not a cocomparability bigraph.* □

Proposition 2.4. *If a bipartite graph G contains an edge-asteroid, then it contains an invertible pair.*

Proof: Suppose that $x_0y_0, x_1y_1, \ldots, x_{2k}y_{2k}$ form an edge-asteroid in G where $x_i \in X$ and $y_i \in Y$ for all i, together with walks joining $x_{i+k}y_{i+k}$ and $x_{i+k+1}y_{i+k+1}$ that contain $x_{i+k}, y_{i+k}, x_{i+k+1}, y_{i+k+1}$ but no vertex adjacent to either of x_i and y_i. Let the walk between $x_{i+k}y_{i+k}$ and $x_{i+k+1}y_{i+k+1}$ be $v_1v_2 \ldots v_t$, where $v_1 = x_{i+k}, v_2 = y_{i+k}$, $v_{t-1} = y_{i+k+1}$, and $v_t = x_{i+k+1}$. Then consider the walk $u_1u_2 \ldots u_t$, with $u_j = x_i$ and $u_{j+1} = y_i$ for each odd j. Since $u_jv_{j+1}, v_ju_{j+1} \notin E(G)$, u_ju_{j+1}, v_jv_{j+1} are independent for each $1 \leq j \leq t-1$. That is, for each $0 \leq i \leq 2k$, there exist congruent walks W_{i+k} and W'_i, where the former walk goes from x_{i+k} to x_{i+k+1}, and the latter walk goes from x_i to x_i. By concatenating the walks $W'_0, W_0, W'_1, W_1, \ldots, W_{k-1}, W'_k$, we obtain a walk from x_0 to x_k, and by concatenating the walks $W_k, W'_{k+1}, W_{k+1}, \ldots, W'_{2k}, W_{2k}$ we obtain a walk from x_k to x_0; these two walks are congruent, and thus x_0, x_k is an invertible pair in G. □

We observe for future reference that the invertible pairs constructed above remain invertible pairs even if the edges $x_0y_0, x_1y_1, \ldots, x_{2k}y_{2k}$ in the edge asteroid are not required to be distinct, as long as there are walks joining $x_{i+k}y_{i+k}$ and $x_{i+k+1}y_{i+k+1}$ that contain $x_{i+k}, y_{i+k}, x_{i+k+1}, y_{i+k+1}$ but no vertex adjacent to either of x_i and y_i. We call such a set of edges a *weak edge-asteroid.*

Proposition 2.5. *If $I(G)$ is a comparability graph, then $\overline{G}$ has a full special orientation that is T-free.*

Proof: Let (X, Y) be a bipartition of G, and let $\prec$ be a transitive orientation of $I(G)$. We orient $\overline{G}$ as follows. Suppose $xy, x'y'$ are two independent edges of G; thus they are adjacent in $I(G)$. Suppose $xy \prec x'y'$ in the transitive orientation of $I(G)$. Then we put $xx' \in \vec{G}$ and $yy' \in \vec{G}$. Note that this will not create a copy of T on x, x', y, y' because in $\vec{G}$ we have the directed edges xx', yy' and the undirected edges $xy', x'y$. Any remaining undirected pairs xx', yy' may be oriented arbitrarily. It remains to show that this is an orientation, i.e., that no edge of $\overline{G}$ inside X or inside Y has been oriented in both directions. Without loss of generality suppose that this happened for an edge xx' inside X; it was oriented from x to x' because $xy \prec x'y'$, and oriented from x' to x because $x'z' \prec xz$, in $I(G)$. Note that all of $xy', xz', x'z, x'y$ are non-edges of G, since $xy, x'y'$ and $xz, x'z'$ are independent pairs of edges. This implies that $I(G)$ also contains edges between xy and $x'z'$ and between xz and $x'y'$. However, $I(G)$ does not contain edges between xy and xz, and between $x'y'$ and $x'z'$, as those pairs intersect and hence are not independent. This contradicts the transitivity of $\prec$. If the edge between xy and $x'z'$ has $xy \prec x'z'$ then by transitivity we would have $xy \prec x'z' \prec xz$, contradicting the fact that there is no edge between xy and xz. If it has $x'z' \prec xy$, then $x'z' \prec xy \prec x'z'$, also a contradiction. □.

Combining Propositions 2.2, 2.4 and 2.5 we obtain the following equivalences. These statements verify that (i) implies (ii), (ii) implies (iii), (iii) implies (iv), and obviously (iv) implies (v). Proposition 1.4 shows that (v) and (vi) are equivalent, and the equivalence of (v) and (i) follows by Gallai's theorem [9].

Theorem 2.6. *The following statements are equivalent for a bipartite graph G:*

(i) $I(G)$ is a comparability graph;

(ii) $\overline{G}$ has a full special orientation that is T-free;

(iii) G does not contain an invertible pair;

(iv) G does not contain a weak edge-asteroid;

(v) G does not contain an edge-asteroid;

(vi) $\overline{I(G)}$ does not contain an asteroid. □

Clearly, Statement (ii) of Proposition 2.1 implies Statement (ii) of Theorem 2.6. In the next section we prove Statement (iii) of Theorem 2.6 implies Statement (ii) of Proposition 2.1. Therefore the statements of Proposition 2.1 and of Theorem 2.6 are all equivalent (see Theorem 3.7).

3 Acyclic orientations

Let G be a bigraph with bipartition (X, Y). The goal of this section is to prove that if G does not contain an invertible pair then $\overline{G}$ has a full special orientation that is acyclic and T-free.

Denote by $\mathcal{F}$ the set of pairs (a, b) where $a \neq b$ and both a, b are in X or both are in Y. Note that $(a, b) \in \mathcal{F}$ if and only if $(b, a) \in \mathcal{F}$. For $(a, b), (f, g) \in \mathcal{F}$, we say that (a, b) *implies* (f, g), and write $(a, b)\Lambda(f, g)$, if there exist congruent (a, f)- and (b, g)-walks. It is easy to verify that Λ is an equivalence relation on $\mathcal{F}$. An equivalence class of this relation will be called an *implication class*. It follows from this definition that there is an implication class which contains both (a, b) and (b, a) if and only if a, b is an invertible pair. Note that $(a, b)\Lambda(f, g)$ if and only if $(b, a)\Lambda(g, f)$.

Two congruent walks

$$a_1a_2 \ldots a_{k-1}a_k \text{ and } b_1b_2 \ldots b_{k-1}b_k$$

are called *standard* if for each $i = 1, 2, \ldots, k-2$ we have $a_i = a_{i+2}$ or $b_i = b_{i+2}$. It is easy to see that if there exist congruent (a, f)- and (b, g)-walks, then there exist standard congruent (a, f)- and (b, g)-walks. Indeed, suppose that for some $i = 1, 2, \ldots, k-2$, we have $a_i \neq a_{i+2}$ and $b_i \neq b_{i+2}$. Note that we must have $a_i \neq b_{i+2}$, since a_i is not adjacent to b_{i+1} in G but b_{i+2} is. Similarly, $b_i \neq a_{i+2}$. So $a_i, a_{i+2}, b_i, b_{i+2}$ are all distinct. Thus the following two walks are congruent:

$$a_1a_2 \ldots a_ia_{i+1}a_ia_{i+1}a_{i+2} \ldots a_{k-1}a_k \text{ and } b_1b_2 \ldots b_ib_{i+1}b_{i+2}b_{i+1}b_{i+2} \ldots b_{k-1}b_k.$$

Continuing this way, we make sure the walks are standard.

For the remainder of this section, let G be a bigraph with bipartition (X, Y) and

$$a_1a_2 \ldots a_{k-1}a_k \text{ and } b_1b_2 \ldots b_{k-1}b_k$$

congruent walks, with an odd k, that begin in X and end in X. Let C_X denote the set of all vertices a_i and b_i with odd i, i.e., all vertices of both paths that lie in X, and let C_Y be defined analogously as the set of all vertices of both paths that lie in Y.

Proposition 3.1. *Suppose $c_1 \in X, c_2 \in Y$ are two vertices satisfying either of the following two conditions:*

- *$c_1c_2 \in E(G)$, but c_1 is not adjacent to any vertex in C_Y and c_2 is not adjacent to any vertex in C_X;*
- *$c_1c_2 \notin E(G)$, but c_1 is adjacent to every vertex in C_Y and c_2 is adjacent to every vertex in C_X.*

Then $(a_1, c_1)\Lambda(a_k, c_1)$ and $(c_1, b_1)\Lambda(c_1, b_k)$.

Proof: Suppose the first condition holds. Then

$$a_1a_2a_3a_4\dots a_{k-1}a_k \text{ and } c_1c_2c_1c_2\dots c_2c_1$$

are congruent walks, thus $(a_1, c_1)\Lambda(a_k, c_1)$. Similarly,

$$c_1c_2c_1c_2\dots c_2c_1 \text{ and } b_1b_2b_3b_4\dots b_{k-1}b_k$$

are congruent walks, thus $(c_1, b_1)\Lambda(c_1, b_k)$.

Suppose now that the second condition holds. Then

$$a_1c_2a_3c_2\dots c_2a_k \text{ and } c_1b_2c_1b_4\dots b_{k-1}c_1$$

are congruent walks, thus $(a_1, c_1)\Lambda(a_k, c_1)$. Similarly,

$$c_1a_2c_1a_4\dots a_{k-1}c_1 \text{ and } b_1c_2b_3c_2\dots c_2b_k$$

are congruent walks, thus $(c_1, b_1)\Lambda(c_1, b_k)$. □

We call a pair $(u, v) \in \mathcal{F}$ *relevant* if its implication class contains at least two pairs. Note that (u, v) is relevant if and only if (v, u) is relevant.

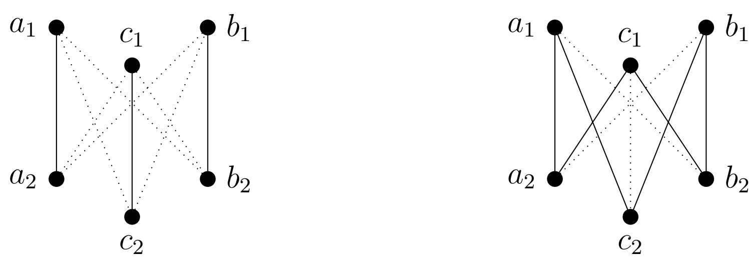

Figure 4: An illustration of the proof of Proposition 3.2

Proposition 3.2. *Suppose in addition that the above walks*

$$a_1a_2\dots a_{k-1}a_k \textit{ and } b_1b_2\dots b_{k-1}b_k$$

are standard.

Suppose $c_1 \in X$ satisfies the following properties:

- *the pair (a_1, c_1) is relevant;*
- *$(a_1, c_1) \not\Lambda(a_1, b_1)$;*
- *$(c_1, b_1) \not\Lambda(a_1, b_1)$.*

Then $(a_1, c_1)\Lambda(a_k, c_1)$ and $(c_1, b_1)\Lambda(c_1, b_k)$.

Moreover, c_1 is either adjacent to both a_{k-1} and b_{k-1} or not adjacent to either of them.

Proof: We will consider two cases: $c_1a_2 \notin E(G)$ and $c_1a_2 \in E(G)$.

Suppose first that $c_1a_2 \notin E(G)$. If there is a vertex $y \in Y$ adjacent to both c_1 and b_1 but not to a_1, then the walks $a_1a_2a_1$ and c_1yb_1 are congruent, contradicting $(a_1, c_1) \not\Lambda (a_1, b_1)$. Therefore every vertex in Y is adjacent to a_1 or is not adjacent to at least one of c_1 and b_1. In particular, we must have $c_1b_2 \notin E(G)$ as b_2 is adjacent to b_1 but not to a_1. Since (a_1, c_1) is relevant, there is a vertex $c_2 \in Y$ such that $c_1c_2 \in E(G)$ and $a_1c_2 \notin E(G)$. It follows that we must have $b_1c_2 \notin E(G)$. Hence the subgraph of G induced by $\{a_1, b_1, c_1, a_2, b_2, c_2\}$ consists of three independent edges, as shown in the left portion of Figure 4.

If c_1 is not adjacent to any vertex in C_Y and c_2 is not adjacent to any vertex in C_X, then the conclusion follows by Proposition 3.1. Therefore we assume that c_1 is adjacent to a vertex in C_Y, or c_2 is adjacent to a vertex in C_X. Let j be the smallest subscript of a vertex in C_Y or in C_X for which this occurs, that is, when j is even, $c_1a_j \in E(G)$ or $c_1b_j \in E(G)$, and when j is odd, $c_2a_j \in E(G)$ or $c_2b_j \in E(G)$. From the above we have $j \geq 3$. Suppose that j is odd. (A similar argument applies when j is even.) Since we have standard walks, $a_{j-2} = a_j$ or $b_{j-2} = b_j$. Assume that $b_{j-2} = b_j$. (Again a similar argument applies when $a_{j-2} = a_j$.) The choice of j implies that we must have $c_2b_j \notin E(G)$ and $c_2a_j \in E(G)$. Since $c_1b_i \notin E(G)$ for each even i, $1 \leq i \leq j$, and $c_2b_i \notin E(G)$ for each odd i, $1 \leq i \leq j$,

$$c_1c_2c_1c_2 \ldots c_2c_1 \text{ and } b_1b_2b_3b_4 \ldots b_{j-1}b_j$$

are congruent walks, hence $(c_1, b_1)\Lambda(c_1, b_j)$. Also,

$$a_1a_2a_3 \ldots a_{j-1}a_jc_2c_1 \text{ and } b_1b_2b_3 \ldots b_{j-1}b_jb_{j-1}b_j$$

are congruent walks, hence $(a_1, b_1)\Lambda(c_1, b_j)$. This implies that $(c_1, b_1)\Lambda(a_1, b_1)$, contradicting our assumption.

Suppose now that $c_1a_2 \in E(G)$. If $c_1b_2 \notin E(G)$, then $c_1a_2a_1$ and $b_1b_2b_1$ are congruent, which implies $(c_1, b_1)\Lambda(a_1, b_1)$, contradicting our assumption. Hence $c_1b_2 \in E(G)$. Since (a_1, c_1) is relevant, there exists a vertex $c_2 \in Y$ which is adjacent to a_1 but not to c_1. If c_2 is not adjacent to b_1, then $a_1c_2a_1$ and $c_1b_2b_1$ are congruent, thus $(a_1, c_1)\Lambda(a_1, b_1)$, again contradicting our assumptions. Therefore $c_2b_1 \in E(G)$. The subgraph of G induced by $\{a_1, b_1, c_1, a_2, b_2, c_2\}$ is C_6 shown in the right portion of Figure 4.

If c_1 is adjacent to every vertex in C_Y and c_2 is adjacent to every vertex in C_X, then the conclusion holds by Proposition 3.1. Let j be the smallest subscript of a vertex in C_Y or in C_X for which, if j is even, $c_1a_j \notin E(G)$ or $c_1b_j \notin E(G)$, and if j is odd, $c_2a_j \notin E(G)$ or $c_2b_j \notin E(G)$. Again from the above we have $j \geq 3$. We again suppose that j is odd. (A similar argument applies when j is even.) We again have $a_{j-2} = a_j$ or $b_{j-2} = b_j$, and assume without loss of generality that $b_{j-2} = b_j$. The choice of j implies that we must have $c_2b_j \in E(G)$ and $c_2a_j \notin E(G)$. Since $c_1a_i \in E(G)$ for each even i, $1 \leq i \leq j$ and $c_2b_i \in E(G)$ for each odd i, $1 \leq i \leq j$, the walks

$$c_1a_2c_1a_4 \ldots c_1a_{j-1} \text{ and } b_1c_2b_3c_2 \ldots b_{j-2}c_2$$

are congruent, whence $(c_1, b_1)\Lambda(a_{j-1}, c_2)$. Also,

$$a_1a_2a_3 \ldots a_{j-1}a_ja_{j-1} \text{ and } b_1b_2b_3 \ldots b_{j-1}b_jc_2$$

are congruent walks, so $(a_1, b_1)\Lambda(a_{j-1}, c_2)$. Hence $(c_1, b_1)\Lambda(a_1, b_1)$, contradicting our assumption. □

We remark that the assumption that (a_1, c_1) is relevant in Proposition 3.2 can be replaced by the assumption that (c_1, b_1) is relevant. This can be seen to be true by switching the roles of a's and b's in the proposition and the proof.

Corollary 3.3. *Let G be a bigraph with bipartition (X, Y). For any distinct vertices $a, b, c \in X$, if (a, b) and (a, c) are relevant but not in the same implication class, then (c, b) is relevant.*

Proof: If (a, b) and (c, b) are in the same implication class, we are done. Otherwise apply Proposition 3.2 with a, b, c playing the roles of a_1, b_1, c_1 respectively. □

Corollary 3.4. *Let G be a bigraph with bipartition (X, Y). Suppose that $a, b, c, d \in X$ with $(a, b)\Lambda(c, d)$ and that one of (a, c) and (c, b) is relevant. Then $(a, c)\Lambda(a, b)$ or $(c, b)\Lambda(a, b)$.*

Proof: Since $(a, b)\Lambda(c, d)$, there exist standard congruent (a, c)- and (b, d)-walks

$$a_1 a_2 \dots a_{k-1} a_k \text{ and } b_1 b_2 \dots b_{k-1} b_k$$

where $a = a_1$, $b = b_1$, $c = a_k$, and $d = b_k$. Suppose for a proof by contradiction that the conclusion of the Corollary does not hold. Then we can apply Proposition 3.2 with a, b, c playing the roles of a_1, b_1, c_1 respectively. But since $c = a_k$, $ca_{k-1} \in E(G)$ and $cb_{k-1} \notin E(G)$, the conclusion of Proposition 3.2 does not hold. This contradiction proves the corollary. □

Corollary 3.5. *Let G be a bigraph with bipartition (X, Y). Suppose that G contains no invertible pair. For any distinct vertices $a, b, c \in X$, if $(c, a)\Lambda(a, b)$, then $(c, b)\Lambda(a, b)$.*

Proof: Applying Corollary 3.4 with a, b, c, a playing the roles of a, b, c, d respectively, we have $(a, c)\Lambda(a, b)$ or $(c, b)\Lambda(a, b)$. If $(c, a)\Lambda(a, b)$, then we cannot have $(a, c)\Lambda(a, b)$ as otherwise a, c are an invertible pair in G, a contradiction to the assumption. Therefore $(c, b)\Lambda(a, b)$. □

Let G be a bigraph and let $\vec{G}$ be a special orientation of $\overline{G}$. (Recall that this means only the edges uv for which $(u, v) \in \mathcal{F}$ are possibly oriented.) We say that $\vec{G}$ is *transitive* if for all u, v, w, if uv and vw are both oriented edges then uw is also an oriented edge. We use $\Lambda(u, v)$ to denote the implication class of $\mathcal{F}$ that contains (u, v). We say that $\vec{G}$ is *closed* if for any $\Lambda(u, v)$, either $\vec{G}$ contains the oriented edge wz for each $(w, z) \in \Lambda(u, v)$ or none of them. Suppose that G contains no invertible pair and that $\vec{G}$ is closed. For an (unoriented) edge uv in $\vec{G}$, we use $\vec{G}(u, v)$ to denote the special orientation of $\overline{G}$ obtained from $\vec{G}$ by orienting edge wz from w to z for each $(w, z) \in \Lambda(u, v)$.

We are now able to prove our main result of this section.

Proposition 3.6. *Let G be a bigraph with bipartition (X, Y). Suppose that G contains no invertible pair. Then $\overline{G}$ has a full special orientation that is acyclic and T-free.*

Proof: We show how to construct a full special orientation that is acyclic and T-free. The construction has two stages. In stage one we orient all edges uv such that (u,v) is relevant (i.e., $\Lambda(u,v)$ has at least two pairs). Such an edge uv must lie either in X or in Y and thus we obtain a special orientation of $\overline{G}$ in stage one. In stage two we extend the special orientation of $\overline{G}$ to a full special orientation of $\overline{G}$.

We proceed with stage one as follows. Initially the special orientation of $\overline{G}$ is $\overline{G}$ itself. Suppose that $\vec{G}$ is a special orientation of $\overline{G}$ that contains an unoriented edge uv such that (u,v) is relevant. If $\vec{G}$ is transitive then we arbitrarily pick an unoriented edge uv such that (u,v) is relevant and extend the special orientation $\vec{G}$ to $\vec{G}(u,v)$. If $\vec{G}$ is not transitive, then there exist u,v,w such that uv and vw are oriented edges but uw is unoriented. Since uv and vw are oriented edges and uw is not, by Corollary 3.5 (u,v) and (v,w) are not in the same implication class. Since (u,v) and (v,w) are both relevant, (u,w) is also relevant according to Corollary 3.3. We extend the special orientation $\vec{G}$ to $\vec{G}(u,w)$. Note that we never orient an edge fg of $\overline{G}$ for which (f,g) is not relevant in stage one.

We show that each special orientation obtained in stage one is acyclic. Assume first that $\vec{G}$ is transitive and that uv is an unoriented edge for which (u,v) is relevant. Suppose that $\vec{G}(u,v)$ contains a directed cycle. Let $C : a_1a_2 \dots a_k$ be the shortest directed cycle contained in $\vec{G}(u,v)$. Since C is the shortest, a_ia_{i+2} is not an oriented edge in $\vec{G}(u,v)$ for each i. In particular, a_ia_{i+2} is not an oriented edge in $\vec{G}$ for each i. It follows that a_ia_{i+1} or $a_{i+1}a_{i+2}$ is not an oriented edge in $\vec{G}$ for each i because $\vec{G}$ is transitive. If neither a_ia_{i+1} nor $a_{i+1}a_{i+2}$ is an oriented edge in $\vec{G}$ for some i, then $(a_i,a_{i+1})\Lambda(u,v)$ and $(a_{i+1},a_{i+2})\Lambda(u,v)$. Hence $(a_i,a_{i+1})\Lambda(a_{i+1},a_{i+2})$ and by Corollary 3.5 $(a_i,a_{i+2})\Lambda(a_{i+1},a_{i+2})$ $(\in \Lambda(u,v))$, a contradiction to the fact that a_ia_{i+2} is not an oriented edge in $\vec{G}(u,v)$. Therefore $\vec{G}$ contains exactly one of the directed edges $a_ia_{i+1}, a_{i+1}a_{i+2}$ for each i. It follows that k is even and at least four. Suppose without loss of generality that $\Lambda(u,v)$ contains (a_1,a_2) and (a_3,a_4). Applying Corollary 3.4 with a_1,a_2,a_3,a_4 playing the roles of a,b,c,d we have that $(a_1,a_2)\Lambda(a_1,a_3)$. This implies that a_1a_3 is also an oriented edge in $\vec{G}(u,v)$. Hence $a_1a_3a_4\dots a_4$ is a directed cycle in $\vec{G}(u,v)$ shorter than C, a contradiction. Assume now that $\vec{G}$ is acyclic but not transitive, which contains oriented edges uv, vw but not uw. Applying Proposition 3.2 with u,w,v playing the roles of a,b,c respectively, we conclude that, for any $(u',w') \in \Lambda(u,w)$, $(u',v)\Lambda(u,v)$ and $(v,w')\Lambda(v,w)$. This means that for any directed edge $u'w'$ in $\vec{G}(u,w)$ but not in $\vec{G}$, $\vec{G}$ contains a directed path from u' to w'. It follows that if $\vec{G}$ is acyclic then so is $\vec{G}(u,w)$. Therefore upon the completion of stage one, we obtain an acyclic special orientation of $\overline{G}$.

Stage two can be carried out as follows. Order the vertices of $\overline{G}$ according to a linear extension of the current $\vec{G}$, and orient all remaining edges in X and in Y to go from the smaller to the larger vertex in this ordering. This gives a full special orientation of $\overline{G}$, which is acyclic and T-free. $\square$

Theorem 3.7. *The following statements are equivalent for a bigraph G.*

(i) G is a cocomparability bigraph;

(ii) $\overline{G}$ *has a full special orientation that is acyclic and* T*-free;*

(iii) $\overline{G}$ *has a full special orientation that is* T*-free;*

(iv) G *does not contain an invertible pair;*

(v) G *does not contain a weak edge-asteroid;*

(vi) G *does not contain an edge-asteroid;*

(vii) $I(G)$ *is a comparability graph.*

Proof: The equivalence of (i) and (ii) is stated in Proposition 2.1; the equivalence of (iii) - (vii) is stated in Theorem 2.6. Together with the fact that (ii) implies (iii), and Proposition 3.6, we conclude that all statements are equivalent. □

4 Further remarks

In addition to the characterization of cocomparability graphs in terms of asteroids, Gallai [9] has given a forbidden subgraph characterization of the graphs. A forbidden subgraph characterization of cocomparability bigraphs can also be obtained from a complete list of minimal bigraphs that contain edge-asteroids given in [8].

As a point made in Section 1, interval containment bigraphs are a better bipartite analogue of interval graphs than interval bigraphs. To support this point of view we state yet another theorem which characterizes interval containment bigraphs, akin to that of Gilmore and Hoffman [10] for interval graphs.

Proposition 4.1. *Each cycle* C_n *with* $n \geq 8$ *contains an edge-asteroid.*

Proof: Denote $C_n : v_1v_2 \ldots v_n$. It is easy to verify that $v_1v_2, v_3v_4, v_4v_5, v_6v_7, v_7v_8$ form an edge-asteroid. □

Combining Propositions 2.4 and 4.1 we have the following:

Corollary 4.2. *If a bipartite graph* G *contains an induced cycle of length* ≥ 8 *then* G *is not a cocomparability bigraph.* □

According to [8, 16], a bipartite graph is an interval containment bigraph if and only if it is a chordal bigraph and has no edge-asteroids. Combining this with Proposition 4.1 and Theorem 3.7, we obtain the following:

Theorem 4.3. *The following statements are equivalent for a bigraph* G*.*

(i) G *is an interval containment bigraph;*

(ii) G *is a chordal cocomparability bigraph;*

(iii) G is a C_6-free cocomparability bigraph. □

Let G be a bipartite graph, with bipartition (X, Y). The *bipartite complement* G' of G has the same vertices as G, and the same bipartition (X, Y), and $xy, x \in X, y \in Y$, is an edge of G' if and only if it is not an edge of G. It follows from Proposition 1.1 that either both G and G' are cocomparability bigraphs or neither is.

The *auxiliary graph* G^+ of G has the vertex set $\mathcal{F}$ in which (u, v) is adjacent to (v, u) and to all (z, w) such that uw and vz are independent edges in G.

Theorem 4.4. *Let G be a bipartite graph with bipartition (X, Y) and let G^+ be the auxiliary graph of G. Then G is a cocomparability bigraph if and only if G^+ is bipartite. Moreover, if G^+ is not bipartite, then any odd cycle of G^+ yields a weak edge-asteroid of G.*

Proof: Suppose that G^+ is bipartite. Let $\mathcal{F}'$ be a colour class of G^+. Then $\mathcal{F}'$ yields a special orientation $\vec{G}$ of $\overline{G}$ such that uv is an oriented edge if and only if $(u, v) \in \mathcal{F}'$. Clearly, $\vec{G}$ is T-free and any full special orientation of $\overline{G}$ that extends $\vec{G}$ is again T-free. Hence G is a cocomparability bigraph by Theorem 3.7.

Conversely, suppose that G is a cocomparability bigraph. Let $\vec{G}$ be a T-free special orientation of $\overline{G}$, and let $\mathcal{F}''$ be the set of pairs corresponding to the oriented edges in $\vec{G}$. Then $\mathcal{F}'' \cap \mathcal{F}$ and $\mathcal{F} \setminus \mathcal{F}''$ form a bipartition of G^+, showing that G^+ is bipartite.

Suppose now that G^+ is not bipartite. Let $(u_0, v_0)(u_1, v_1) \cdots (u_{2k}, v_{2k})(u_0, v_0)$ be an odd cycle in G^+. By the definition of G^+, u_i and v_i are both in X or in Y for each i. Consequently, there must exist some j such that $u_j, v_j, u_{j+1}, v_{j+1}$ are all in X or in Y, in which case $u_j = v_{j+1}$ and $v_j = u_{j+1}$. Without loss of generality assume that $j = 2k$ (i.e., $u_{2k} = v_0$ and $v_{2k} = u_0$).

We claim that the following $4k - 1$ edges

$$u_0u_1, u_1u_2, \ldots, u_{2k-1}u_{2k}, v_1v_2, v_2v_3, \ldots, v_{2k-1}v_{2k}$$

form a weak edge-asteroid in G. Indeed, for i, $v_iv_{i+1}v_{i+2}$ is a path joining v_iv_{i+1} and $v_{i+1}v_{i+2}$ containing no vertex adjacent either of u_i, u_{i+1}, and $u_iu_{i+1}u_{i+2}$ is a path joining u_iu_{i+1} and $u_{i+1}u_{i+2}$ containing no vertex adjacent either of v_i, v_{i+1}. Moreover, $u_{2k-1}u_{2k}v_2$ is a path joining $u_{2k-1}u_{2k}$ and v_1v_2 containing no vertex adjacent to either of v_{2k-1}, v_{2k}, and $v_{2k-1}v_{2k}u_2$ is a path joining $v_{2k-1}v_{2k}$ and u_1u_2 not containing no vertex adjacent to either of u_{2k-1}, u_{2k}. Hence G contains a weak edge-asteroid and by Theorem 3.7 is not a cocomparability bigraph. □

Corollary 4.5. *There is a polynomial time algorithm to decide whether a bigraph G is a cocomparability bigraph. Moreover, the algorithm finds a full special orientation of $\overline{G}$ that is T-free if G is a cocomparability bigraph, or else it exhibits a weak edge-asteroid to certify that G is not a cocomparability bigraph.* □

References

[1] R.P. Anstee and M. Farber, Characterizations of totally balanced matrices, J. Algorithms 5 (1984) 215 - 230.

[2] A. Brandstädt, V.B. Le, and J. Spinrad, **Graph Classes**, SIAM Monographs on Discrete Math.and Appl., 1999.

[3] D.E. Brown, S.C. Flink, J.R. Lundgren Characterizations of interval bigraphs and unit interval bigraphs, Congr. Numerantium 157 (2002) 79 - 93.

[4] V. Chvátal, Which claw-free graphs are perfectly orderable? Discrete Applied Math. 44 (1993) 39 - 63.

[5] D.G. Corneil, S. Olariu and L. Stewart, Asteroidal triple-free graphs, SIAM J. Discrete Math. 10 (1997) 399 - 430.

[6] P. Damaschke, Forbidden ordered subgraphs, Topics in Combinatorics and Graph Theory (1990) 219 - 229.

[7] V. Deineko, R. Rudolf, and G.J. Woeginer, A general approach to avoiding two by two submatrices, Computing 52 (1994) 371 - 388.

[8] T. Feder, P. Hell, and J. Huang, List homomorphisms and circular arc graphs, Combinatorica 19 (1999) 487 - 505.

[9] T. Gallai, Transitiv orientierbare Graphen, Acta Mathematica Academiae Scientiarum Hungarica 18 (1967) 25 - 66.

[10] P.C. Gilmore and A.J. Hoffman, A characterization of comparability graphs and interval graphs, Canad. J. Math. 16 (1964) 539 - 548.

[11] M.C. Golumbic, **Algorithmic Graph Theory and Perfect Graphs**, Academic Press (1980).

[12] M.C. Golumbic and C.F. Goss, Perfect elimination and chordal bipartite graphs, J. Graph Theory 2 (1978) 155 - 163.

[13] F. Harary, J.A. Kabell, F.R. McMorris, Bipartite intersection graphs, Comment. Math. Univ. Carolin. 23 (1982) 739 - 745

[14] P. Hell and J. Huang, Interval bigraphs and circular arc graphs, J. of Graph Theory 46 (2004) 313 - 327.

[15] A.J. Hoffman, M. Sakarovich and A.W. Kolen, Totally balanced and greedy matrices, SIAM J. Algebraic Discrete Methods 6 (1985) 721 - 730.

[16] J. Huang, Representation characterizations of chordal bipartite graphs, J. Combinatorial Theory B 96 (2006) 673 - 683.

[17] J. Huang, Non-edge orientation and vertex ordering characterizations of some classes of bigraphs, Discrete Applied Math. 245 (2018) 190 - 193.

[18] A.J. Hoffman, A.W. Kolen, and M. Sakarovitch, Totally-balanced and greedy matrices, SIAM J. Algebraic and Discrete Methods 6 (1985) 721 - 730.

[19] B. Klinz, R. Rudolf, and G.J. Woeginger, Permuting matrices to avoid forbidden submatrices, Discrete Applied Math. 60 (1995) 223 - 248.

[20] V.B. Le, On opposition graphs, coalition graphs, and bipartite permutation graphs, Discrete Applied Math. 168 (2014) 26 - 33.

[21] C.G. Lekkerkerker and J.Ch. Boland, Representation of a finite graph by a set of intervals on the real line, Fund. Math. 51 (1962) 45 - 64.

[22] A. Lubiw, Γ-free matrices, Master's thesis, University of Waterloo, 1982.

[23] A. Lubiw, Doubly lexical orderings of matrices, SIAM J. Computing 16 (1987) 854 - 879.

[24] R.M. McConnell, Linear-time recognition of circular-arc graphs, Algorithmica 37 (2003) 93–147.

[25] R.M. McConnell and J.P. Spinrad, Modular decomposition and transitive orientation, Discrete Math. 201 (1999) 189 - 241.

[26] H. Müller, Recognizing interval digraphs and interval bigraphs in polynomial time, Discrete Appl. Math. 78 (1997) 189 - 205. Erratum, hhp://www.comp.leeds.ac.uk/hm/publ.html.

[27] R. Paige and R.E. Tarjan, Three partition refinement algorithms, SIAM J. Computing 16 (1987) 973 – 989.

[28] M. Sen, S. Das, A.B. Roy, and D.B. West, Interval digraphs: An analogue of interval graphs, J. Graph Theory 13 (1989) 189 - 202.

[29] M. Sen, B.K. Sanyal, and D.B. West, Representing digraphs using intervals or circular arcs, Discrete Math 147 (1995) 235 - 245.

[30] A.M.S. Shrestha, S. Tayu and S. Ueno, On orthogonal ray graphs, Discrete Applied Math 158 (2010) 1650 - 1659.

[31] J.P. Spinrad, A. Brandstädt, and L. Stewart, Bipartite permutation graphs, Discrete Applied Math. 18 (1987) 279 - 292.