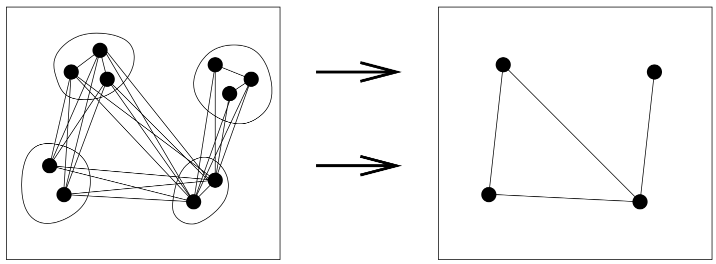
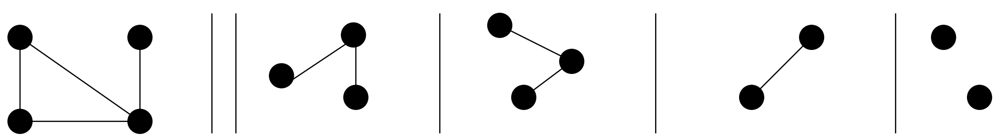
Class exercises on transitive orientation, CS620, McConnell, Spring '08

Revised 5/12 14:10; these exercises are helpful as a study guide for the final, and at the review session we will go over any you want to see solved.

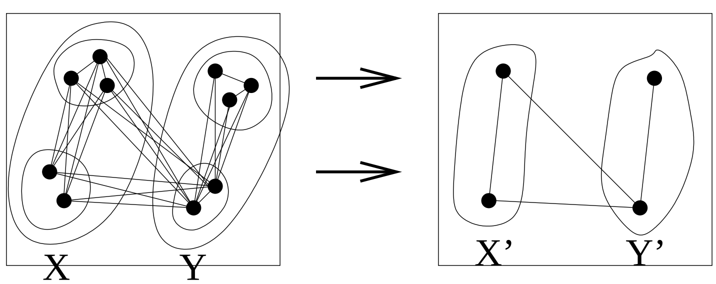
1. Because two disjoint modules are either completely nonadjacent or completely adjacent, a partition of G into modules gives a quotient graph:

   G may be reconstructed from the quotient and the subgraphs induced by the modules (the "factors") by "substituting" the factors into the quotient:

   Suppose A is the set of vertices of G that map to a set A' of vertices of the quotient. Show that A is a module of G if and only if A' is a module of the quotient:

   (Both X and X' are modules and neither Y nor Y' are modules.)

   Let's call this the *quotient rule*.

2. Show how to draw edges between children of a node X in the modular decomposition tree to create a quotient on $G[X]$. Show how you can reconstruct G using only the quotients on children of nodes in the modular decomposition tree.

3. In the case where G is disconnected, what edges do you have to draw? How about the case where the complement $\overline{G}$ of G is disconnected?

4. Every graph has V and its one-element subsets as modules. A graph is *prime* if these are its only modules. Why must the quotient on children of a Case 3 node (G is connected and so is $\overline{G}$) be a prime graph?

5. Show that no Case 1 node can be a child of another, and no Case 2 node can be a child of another.

6. Suppose ab and bc are two undirected edges in a comparability graph and a and c are nonadjacent. Show that in any transitive orientation, either (a, b) and (c, b) are both directed edges or (b, a) and (b, c) are both directed edges.

7. Following the observation from the previous point, let us say that two edges *directly force* each other if they share an endpoint and the other two endpoints are nonadjacent. Show that any *acyclic* orientation of G such that no direct forcing constraint is violated must be a transitive orientation.

8. Show that if X is a module, the complement of $G[X]$ is connected, and z is a vertex outside of X, then in any transitive orientation, either all edges between z and X are oriented toward z or they are all oriented toward X.

9. Using this, show that if X and Y are modules and the complements of $G[X]$ and $G[Y]$ are connected, then either all edges between X and Y are oriented toward Y or they are all oriented toward X.

10. Show that a graph that is an independent set has exactly one transitive orientation, and a graph that is complete has exactly $n!$ transitive orientations.

11. Let us say that they *indirectly force* each other if one of them forces an edge that forces an edge that forces an edge ... that forces the other. Let a *color class* on the edges be a set of edges that indirectly force each other.

    (a) **A color class ends at a module boundary.** Show that if X is the set of vertices spanned by a color class, then X is a module. *Hint: Assume a spoiler and show that it is incident to an edge that is forced by the color class, hence not outside the set spanned by the color class and therefore not a spoiler.*

    (b) **A color class can't cross a module boundary.** Let us say that an edge e is *contained* in vertex set X if *both* endpoints of e are contained in X. Show that if X is a module and e_1 is contained in X and e_2 is not, then e_1 does not even indirectly force e_2.

12. Show that a color class of a prime graph must span all of the vertices.

13. It turns out that not only do all color classes of a prime graph span all vertices, but a prime graph only has one color class. (We ran out of time to prove this, though if you understood the algorithm for transitively orienting a prime graph, you will see that it also serves as a proof.) Using this, figure out how many transitive orientations a prime graph has.

14. Using any of the previous points, show that if X and Y are children of a Case 3 node, then in any transitive orientation, either all edges are pointed from X to Y or all edges are point from Y to X. (*Hint: Show that this is forced by a transitive orientation of the quotient on the children of the parent.*)

15. Show how to use the modular decomposition to determine the number of transitive orientations of a comparability graph if the previous points are true.