Class exercises, 5/1, CS620, McConnell, Spring '08 (to be continued on 5/6)

1. Discover a representation of all the modules of an undirected graph, $G = (V, E)$

   Let us say that two sets X and Y *properly overlap* if they intersect but neither contains the other. If $X \subset V$ and $y \in V - X$, then y is a *spoiler for* X if X contains both a neighbor of y and a non-neighbor of y. A *module* of G is a nonempty subset of V that has no spoilers.

   (a) Show that if two modules X and Y are disjoint, then either every element of $X \times Y$ is an edge, or none of them is. **(Mike solved this.)**

   (b) Show that when two modules X and Y of G properly overlap, $X \cup Y$ is also a module. **(Nate solved this.)**

   (c) Show that when two modules X and Y of G properly overlap, $X \cap Y$ is also a module.

   (d) Show that when two modules X and Y of an undirected graph properly overlap, then $X \Delta Y = (X - Y) \cup (Y - X)$ is also a module. **(Nissa solved this.)**

   (e) Show that at most one of a graph and its complement can be disconnected. **(Weston solved this.)**

   (f) Show that if a graph is disconnected and the connected components are $\{X_1, X_2, \ldots, X_k\}$, then every union of connected components is a module, and all other modules are subsets of some X_i. **(Artem solved this.)**

   (g) Show that the modules of a graph are the same as the modules of its complement.

   (h) Show that if both a graph and its complement are connected, then the maximal modules that are proper subsets of the set V of vertices are a partition $\{X_1, X_2, \ldots, X_k\}$ of V. (That is, every member of V is a member of exactly one X_i.)

   (i) Show that in this last case, every module of G is either V, X_i for some i from 1 through k, or a subset of some X_i.

   (j) If $\emptyset \subset X \subseteq V$, let $G[X]$ denote the subgraph of G induced by X. Show that if X_i is a module, then the modules of G that are subsets of X_i are exactly the modules of $G[X_i]$.

2. This gives the following representation for the modules of a graph, called the *modular decomposition*. It is a tree whose nodes are sets of vertices.

   As a base case, if the graph has one vertex, v.

   Otherwise, consider the following cases. Case 1: If the graph is disconnected, let $\{X_1, X_2, \ldots, X_k\}$ be the connected components. Case 2: If the complement of the graph is disconnected, let $\{X_1, X_2, \ldots, X_k\}$ be the connected components of the complement. Case 3: Otherwise, let $\{X_1, X_2, \ldots, X_k\}$. The children of V are the roots of the modular decompositions of $G[X_1]$, $G[X_2]$, ..., $G[X_k]$, and V carries a label that tells which of the three cases it falls under. This tree is called the *modular decomposition* of the graph.

(a) Using some of the things you proved above, show that a graph is a module if and only if it is a node of this tree, or a union of children of a node labeled Case 1 or Case 2.

(b) For each node of the tree, we want to be able to support the operation of returning the set of vertices that the node represents in time proportional to the size of that set. One way to do this is to represent each node of the tree with a linked list of vertices.

   i. Show that the tree takes $\Theta(n^2)$ space in the worst case if you represent it this way.

   ii. Give an $O(n)$-space representation of the tree that supports the operation just as fast.