Elementary Landscapes of Frequency Assignment Problems

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Abstract

We analyze various forms of the Frequency Assignment Problem using the theory of elementary landscapes. We show that three variants of the Frequency Assignment Problem are either directly an Elementary Landscape, or are a superposition of two Elementary Landscapes. We also examine the computability of neighborhood averages for partial neighborhoods.

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1. INTRODUCTION

We will define a landscape for a combinatorial problem using a triple \((X, N, f)\), where \(f : X \rightarrow \mathbb{R}\) defines the objective function and the neighborhood operator function \(N(x)\) generates the set of points reachable from \(x \in X\) in a single application of the neighborhood operator. If \(y \in N(x)\) then \(y\) is a neighbor of \(x\). The landscape that is induced can be used as a search space for optimization using local search methods. Without loss of generality we can define \(f\) so as to be either minimized or maximized over \(X\).

Grover [5] first showed that certain problems (including the Traveling Salesman Problem and Graph Coloring) have common and natural local search neighborhoods that can be modeled using a wave equation. Stadler [8] named this class of problems “elementary landscapes” and has explored various properties of elementary landscapes.

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Elementary landscapes are induced by a neighborhood operator that is characterized by a wave equation [5]:

\[
\overline{\text{avg}}\{f(y)\} = f(x) + \frac{k}{d} (\overline{f} - f(x))
\]

where \(d = |N(x)|\) is the size of the neighborhood which we assume is the same for all the solutions in the search space, \(\overline{f}\) is the average solution evaluation over the entire search space, and \(k\) is a characteristic constant of \(f\). The function \(f\) of an elementary landscape is an eigenvector of the Laplacian matrix of the neighborhood (up to an additive constant), and the characteristic constant \(k\) associated with \(f\) is the additive inverse of the corresponding eigenvalue \(\lambda\).

For all elementary landscapes it is possible to compute \(\overline{f}\), the average solution evaluation over the entire search space, with a small number of evaluations of \(f\) or no evaluation at all. The wave equation also makes it possible to compute the average value of the fitness function \(f\) evaluated over all of the neighbors of \(x\); we denote this average using \(\overline{\text{avg}}\{f(y)\}\):

\[
\overline{\text{avg}}\{f(y)\} = \frac{1}{|N(x)|} \sum_{y \in N(x)} f(y)
\]

Other properties of the elementary landscapes also follow. Assuming \(f(x) \neq \overline{f}\) then

\[
f(x) < \min \left\{ \overline{\text{avg}}\{f(y)\}, \overline{f} \right\} \quad \lor \quad f(x) > \max \left\{ \overline{\text{avg}}\{f(y)\}, \overline{f} \right\}
\]

This implies that all maxima are greater than \(\overline{f}\) and all minima are less than \(\overline{f}\) [2].

Whitley and Sutton [11] use a component model to explain why certain classes of elementary landscapes obey the wave equation. This involves decomposing the objective function \(f\) into a linear combination of cost components. The cost components sometimes take the form of a cost matrix; for example, the distance matrix that is used to calculate the distance between cities in the Traveling Salesman Problem is such that each distance is one component of the cost function. In Graph Coloring, the components can also be defined to be the weights of a lower triangular cost matrix \(M\), where \(m_{ij} = 0\) when \(i > j\). In Graph Coloring the cost is usually 1 for each conflicted edge; however, we show the Graph Coloring problem is still elementary when an arbitrary weight matrix is used to assign costs to conflicted edges. This sets the stage for looking at several variants of the Frequency Assignment Problem [6].
Whitley and Sutton [11] present a formula that computes averages for partial neighborhoods for Graph Coloring. In this paper the theorem and proof is generalized to cover the Basic Frequency Assignment Problem and a wider range of partial neighborhoods. We also correct an omission in their proof, which fails to explicitly consider some forms of the graph coloring problem.

In the next section, we introduce the component model for Elementary Landscapes. In Section 3 we analyze three forms of the Frequency Assignment Problem. First, we cover the Basic FAP that is equivalent to the Weighted Graph Coloring problem. We then look at more complex forms of the Frequency Assignment Problem and analyze partial neighborhoods for the Weighted Graph Coloring. Finally, we conclude in Section 4.

2. THE COMPONENT MODEL

Let \( C \) represent the set of components that make up the cost function. In the Traveling Salesman Problem these are the distances between cities. In the Weighted Graph Coloring, given some coloring of a graph \( G \) with vertices \( V \) and edges \( E \), we define a matrix \( M \) such that entry \( m_{ij} (i \leq j) \), stores the cost that is incurred if vertices \( v_i \) and \( v_j \) are assigned the same color. The weights of \( M \) are the components that make up the set of components in \( C \).

For any incumbent search point \( x \), we will separate the set of components denoted by \( C \) into those that contribute to \( f(x) \) and those components that do not contribute to \( f(x) \). For the Weighted Graph Coloring problem, a solution \( x \) has a cost \( f(x) \) that is a summation over a subset of the cost matrix. Thus, the cost matrix can be partitioned into those weights that contribute to \( f(x) \) and those weights that do not contribute to \( f(x) \).

We can also characterize a neighbor \( y \in N(x) \) in terms of the components that it shares with \( x \) and the components in \( y \) that are not found in \( x \). For the wave equation to hold, the components in \( x \) and the components in \( N(x) \) must together include all of the components in \( C \). By a slight abuse of notation, we will let \( (C - x) \) refer to components in \( C \) that do not appear in solution \( x \) and thus do not contribute to \( f(x) \). When transforming \( x \) to some \( y \in N(x) \) we subtract a subset of components from \( x \) and add new components from \( (C - x) \). All components in \( x \) are uniformly sampled for removal from \( x \) as \( N(x) \) is constructed. All components in \( (C - x) \) are uniformly sampled in the construction of the set of neighbors \( N(x) \). We will assume the neighborhood size is regular and denoted by \( d \).

There are also 3 ratios \( p_1, p_2 \) and \( p_3 \) that are used in the component model.

\[
\text{avg}\{f(y)\} = f(x) - p_1 f(x) + p_2 (\sum_{c \in C} c - f(x))
\]

and

\[\sum_{c \in C} c = \bar{f} / p_3\]

where we use \( c \) to represent both a component and its value. The ratios \( p_1 \) and \( p_2 \) can be defined either in relationship to a single move in the neighborhood, or in relationship to the entire neighborhood. Thus, \( 0 < p_1 < 1 \) is the proportion of components that contribute to the evaluation of \( f(x) \) and that are excluded when a single move is made to a single neighboring point; but alternatively \( p_1 = \frac{\alpha}{d} \) where \( \alpha \) is the number of times a component in solution \( x \) is removed when all of the neighbors are considered, and \( d \) is the neighborhood size. Similarly, \( 0 < p_2 < 1 \) is the proportion of components in \( (C - x) \) that are included in solutions that make up the neighborhood of \( x \); alternatively, \( p_2 = \frac{\beta}{d} \) where \( \beta \) is the number of times a component in \( C - x \) is included in a solution of the neighborhood.

Finally, \( 0 < p_3 < 1 \) is the proportion of the total components in \( C \) that contribute to the cost function for any randomly chosen solution; \( p_3 \) is independent of the neighborhood size. This means that \( \bar{f} \) and the components are related as follows.

\[
\bar{f} = p_3 \sum_{c \in C} c \quad \text{and therefore} \quad \sum_{c \in C} c = \bar{f} / p_3
\]

The following theorem then holds by simple algebra [11].

**Theorem 1.** If \( p_1, p_2 \) and \( p_3 \) can be defined for any regular landscape such that the evaluation function can be decomposed into components where \( p_1 = \frac{\alpha}{d} \) and \( p_2 = \frac{\beta}{d} \) and

\[
\bar{f} = p_3 \sum_{c \in C} c = \frac{\beta}{\alpha + \beta} \sum_{c \in C} c
\]

then the landscape is elementary and

\[
\text{avg}\{f(y)\} = f(x) + \frac{\alpha + \beta}{d}(\bar{f} - f(x))
\]

3. FREQUENCY ASSIGNMENT

We will consider 3 different forms of the Frequency Assignment Problem (FAP).

1. The Basic Frequency Assignment Problem is defined to be exactly the same as the Weighted Graph Coloring problem. Conflicts occur when connected vertices (those with non-zero cost) are assigned the same frequency. We will denote this function by \( f_s \). We analyze this problem in the next section.

2. The Symmetric Adjacent Frequency Assignment Problem. We will denote this function by \( f_s \). In the frequency assignment problem there is a concept of nearest that does not exist in the graph coloring problem. Given an ordered set of frequencies \( q_1 \) to \( q_5 \) and a pair of vertices \( v_i \) and \( v_j \) which are connected by a non-zero cost edge \( e_{i,j} \) we can define two types of conflicts: 1) direct conflicts, where \( v_i \) and \( v_j \) are assigned the same frequency and 2) adjacent conflicts, where \( v_i \) and \( v_j \) are assigned frequencies \( q_k \) and \( q_{k+1} \). The Symmetric Adjacent Frequency Assignment Problem is non-standard in two ways. First, it only counts the adjacent conflicts (the direct conflicts are evaluated separately by function \( f_b \)), and second, we will consider the lowest frequency, \( q_1 \), and the highest one, \( q_5 \), also to be a source of adjacent conflict. This makes the set of conflicts symmetric; every frequency conflicts with exactly two other frequencies. We analyze this problem in Section 3.2.

3. The Asymmetric Adjacent Frequency Assignment Problem will look at adjacent conflicts under the more common assumption that the highest and lowest frequencies, \( q_1 \) and \( q_5 \), are not a source of conflict; it will also include the direct conflicts in the evaluation function. This problem is analyzed in Section 3.3.
3.1 The Basic FAP

We define the Basic FAP to be a Weighted Graph Coloring problem. We then construct our proofs in terms of the Weighted Graph Coloring problem. The concept of color assignment and frequency assignment can be interchanged because only direct conflicts are considered.

Let $G$ be a graph, $V$ the set of vertices, and $E$ the set of edges. In order to be as general as possible, we will assume graph $G$ is fully connected. We define a lower triangular cost matrix $M$ that defines a cost associated with every pair of vertices. Thus, given $|V|$ vertices the cost matrix $M$ is defined over the lower triangle of a $|V| \times |V|$ matrix. If we wish to consider a graph $G'$ that is not fully connected, this can be transformed to another graph $G$ that is fully connected, but where the cost for an edge that does not appear in $G'$ is set to zero.

Given $r$ colors, the goal is to assign a color to every vertex of graph $G$ so as to minimize the total cost of the coloring.

In the basic graph coloring problem, a conflicting coloring is assigned a cost 1. However this is really just a special case of the general Weighted Graph Coloring problem. In the Basic FAP, we can think of the colors as a label for the frequencies that are assigned to vertices. The cost of assigning a color to a vertex is incurred when vertices $v_i$ and $v_j$ have the same color (frequency).

A candidate solution $x$ colors every vertex in $G$. In formal terms, a solution is a map from $V$ to the set of possible colors $\{1, 2, \ldots, r\}$. For every pair of vertices $v_i$ and $v_j$ in $V$, the evaluation function $f(x,v_i,v_j)$ adds the cost $m_{ij} \in M$ if $v_i$ and $v_j$ are the same color, that is, $f(x,v_i,v_j) = f(v_i,v_j)$. This means that every non-zero weight in matrix $M$ either 1) contributes to the cost function $f(x,v_i,v_j)$ or 2) does not contribute to the cost function $f(x,v_i,v_j)$.

We say that two solutions $x, y$ are neighbors if there exists a vertex $v \in V$ such that $x(v)$ is a neighbor of $y(v)$ and for the remaining vertices $w \in V$ we have $x(w) = y(w)$. The neighborhood operator $N$ recolors every vertex; since the vertex has a color, there are $r - 1$ recolorings of the vertex. Since there are $|V|$ vertices, and each vertex can be recolored in $r - 1$ ways, the size of the neighborhood is $|V|(r-1)$.

For every edge $e_{i,j}$ vertex $v_i$ can be colored in $r$ ways, and vertex $v_j$ can be colored in $r$ ways. This yields $r^2$ possible assignments per edge, but only $r$ of these produce a conflict. Thus, each edge yields a conflict in $r/r^2$ colorings which implies $p_3 = 1/r$. Since this is uniformly true for every edge, the average cost over all solutions is given by

$$p_3 = \frac{2}{|V|(r-1)} = \frac{\beta}{d}$$

Again $p_1$ and $p_2$ are independent of the cost matrix used. Adding the terms to the component model yields:

$$\text{avg}\{f_s(y)\} = f_s(x) - p_1 f_s(x) + p_2 (f_3/p_3 - f_s(x))$$

$$= f_s(x) + \frac{2r}{|V|(r-1)} (f_3 - f_s(x))$$

This form satisfies Grover’s wave equation with characteristic constant $k = 2r$ and neighborhood size $d = |V|(r-1)$. We use $f^c(x)$ and $f_3$ are independent of the sampling rates of the components.

3.2 Symmetric Adjacent FAP

In the Frequency Assignment Problem the “colors” are actually an ordered set. A color (or frequency) $q_k$ is said to be adjacent to two other colors (or frequencies) $q_{k-1}$ and $q_{k+1}$ unless $q_k$ is the highest or lowest color frequency. If the frequencies are numbered from 1 to $r$, then we can classify frequency $q_1$ and $q_r$ as extreme-frequencies and all of the other frequencies are internal-frequencies. We will sometimes continue to refer to the frequencies as “colors”.

In the Adjacent FAP, we avoid assigning connected vertices adjacent frequencies as well as the same frequency. Thus, when “recoloring” a vertex $v_i$, there can be 3 possible conflicts with another vertex $v_j$: if $v_j$ has frequency $q_k$, then a conflict occurs if $v_i$ is color $q_k$ or $q_{k+1}$ or $q_{k-1}$. The existence of a highest and lowest frequency creates an asymmetry in terms of recoloring particular edges. Before tackling the asymmetric case, we will first create a symmetric version of the problem. We will assume that the highest and lowest frequencies $q_1$ and $q_r$ are also adjacent. We call this the Symmetric Adjacent Frequency Assignment Problem, and denote the fitness function by $f_s$.

Also, in the Symmetric Adjacent Frequency Assignment problem we will only consider the adjacent conflicts (not the direct conflicts). Thus, if $v_j$ has frequency $q_k$, then a conflict occurs if $v_i$ is assigned $q_{k+1}$ or $q_{k-1}$; we deal with the direct conflicts separately. This will set the stage for modeling the more common asymmetric adjacent frequency assignment problem.

The Symmetric Adjacent FAP is just a modified form of Weight Graph Coloring where each vertex coloring conflicts with 2 other colorings.

We again assign $r$ frequencies (i.e., colors) to the graph. Use the cost matrix $M'$ to assign a cost to a solution $x$ if adjacent frequencies are assigned to two connected (non-zero cost) vertices. We use $M'$ instead of $M$ because, in general, the costs of adjacent conflicts are different from the costs of direct conflicts. We again seek an assignment that has minimal cost. We first define $p_1$ and $p_2$ for this new problem.

Consider two vertices $v_1$ and $v_2$ that are in conflict. When one vertex is assigned new frequencies there are $r - 2$ frequencies that do not cause an adjacent conflict and one that
causes conflict. Across all neighbors $|V|/|(r-1)|$, we reassign both vertices with each of the $r-2$ frequencies, and then we have $2(r-2)$ neighbors without that conflict. Therefore it follows that:

$$p_1 = 2(r-2)/(|V|/|(r-1)|) = \frac{\alpha}{d}$$

As to $p_2$, given a vertex $v_1$ and $v_2$ that are connected by an edge and are not currently in conflict, $v_1$ can be colored in two ways that conflict with $v_2$, and $v_2$ can be colored in two ways that conflict with $v_1$. Therefore it follows that:

$$p_2 = 4/(|V|/|(r-1)|) = \frac{\beta}{d}$$

Finally, we consider the calculation of $f_s$. If we generate a random solution and consider any single edge there are $r^2$ colorings for that edge that occur uniformly across the search space. Without loss of generality, assume we color a “left” vertex first, and then color the “right” vertex; the left vertex is assigned one of $r$ colors, and each color conflicts with 2 colorings of the right vertex. Thus, $p_1$ which calculates the proportion of colorings that yield a conflict for each component, is given by $2r/r^2 = 2/r$.

Adding the terms in the component model formula yields:

$$\text{avg}\{f_s(y)\} = f_s(x) - p_1 f_s(x) + p_2 (f_s/p_1 - f_s(x)) = f_s(x) + \frac{2r}{|V|/|(r-1)|} (f_s - f_s(x))$$

The resulting elementary landscape yields the same characteristic constant $k$ that characterizes the direct conflict evaluation function. In fact, it is easy to show that whether a color conflicts with 1 other color (as in $f_b$) or conflicts with 2 other colors (as in $f_s$) or conflict with $g$ colors ($g < r$) makes no difference: the characteristic constants are the same. Thus, we will define the combined fitness function $f_{bs} = f_b + f_s$

which considers both the direct conflicts and the symmetric adjacent conflicts. The sum of two functions that are elementary with the same characteristic constants is also elementary with the same constant. Thus, the new function $f_{bs}$ is elementary where

$$\text{avg}\{f_{bs}(y)\} = f_{bs}(x) + \frac{2r}{|V|/|(r-1)|} (f_{bs} - f_{bs}(x))$$

### 3.3 The Asymmetric Adjacent FAP

Now let us consider the Asymmetric Adjacent Frequency Assignment Problem. In this case, adjacent conflicts yield a cost, but the extreme frequencies, $q_i$ and $q_r$, are not adjacent. We first define a function $A$ that classifies pairs of frequencies according to their type. The function $A$ can take the following four values:

$$A(u, t) \in \{\text{unequal extreme, equal extreme, internal, internal + extreme}\}$$

where “internal” denotes both $u$ and $t$ are (internal) frequencies which are not extreme frequencies. The evaluation “equal extreme” denotes both $u$ and $t$ are the same extreme frequency. The evaluation “unequal extreme” denotes $u$ and $t$ are two different extreme frequencies. The evaluation “internal + extreme” denotes that only one of $u$ or $t$ is an extreme frequency (the other one is an internal frequency).

The fitness function for the Asymmetric Adjacent FAP, which we will denote by $f_{aa}$, can be written as the sum of two functions:

$$f_{aa}(x) = f_{ba}(x) - f_{if}(x)$$

where $f_{ba} = f_b + f_s$ is the function defined in the previous section. The new function $f_{if}$ is defined as:

$$f_{if}(x) = \sum_{e_{i,j} \in E} m_{ij} \phi_{i,j}(x)$$

$$\phi_{i,j}(x) = \begin{cases} 1 & \text{if } A(v_i, v_j) = \text{unequal extreme} \\ 0 & \text{otherwise} \end{cases}$$

where $m_{ij}$ is the adjacent cost assigned to edge $e_{i,j}$ when vertices $v_i$ and $v_j$ are assigned adjacent frequencies. The reader should not confuse this cost with $m_{ij}$, which is the direct cost assigned to edge $e_{i,j}$ when vertices $v_i$ and $v_j$ are assigned the same frequency. The new function takes value $m_i'$, when the vertices $v_i$ and $v_j$ are colored with different extreme frequencies (the “’f” characters in the function name stands for extreme frequencies). The function $f_{ba}$ counts different extreme frequencies as adjacent conflict. Therefore we subtract the corresponding cost associated with extreme frequencies in Equation (1).

In the previous sections we have shown $f_b$ and $f_s$ are elementary landscapes in the considered neighborhood. Furthermore, the characteristic constant associated to these elementary landscapes is the same ($k = 2r$), and, therefore, the combined function $f_{bs}$ is also an elementary landscape with the same characteristic constant. Thus, we just need to focus on the new $f_{if}$ function.

We will not use the component model for the analysis of $f_{if}$, instead we will use a characterization of elementary landscapes, namely: a fitness function is elementary in a given neighborhood if and only if there is a linear relationship between the average value of the fitness function in the neighborhood of one solution and the fitness value of the solution, that is, if there exist two constants $a$ and $b$ such that the equation

$$\text{avg}\{f(y)\} = af(x) + b$$

holds for each solution $x$. We will use this characterization of the elementary landscapes to prove that the function $\phi_{i,j}(x)$ is the sum of two elementary landscapes. Before proving this, we need to define a new auxiliary function:

$$\phi_{i,j}(x, z) = \begin{cases} z & \text{if } A(v_i, v_j) = \text{unequal extreme} \\ -1 & \text{if } A(v_i, v_j) = \text{equal extreme} \\ 1 & \text{if } A(v_i, v_j) = \text{internal} \\ 0 & \text{if } A(v_i, v_j) = \text{internal + extreme} \end{cases}$$

We can compute the average contribution of any edge in the function $\phi_{i,j}(x, z)$ from first principles. The average is independent of which edge is considered, but it does depend on the assignment of $z$; thus the average will be denoted by $\bar{\phi}_{i,j}$. Of the $r^2$ possible colorings of two adjacent vertices, 2 have unequal extreme frequencies, 2 have equal extreme frequencies, and $(r - 2)^2$ have internal frequencies; in the remaining cases the function has value 0. Thus, we can write:
The most important result related to the $\phi_{i,j}$ functions is the following

**Lemma 1.** Using the standard “recoloring” neighborhood, the function $\phi_{i,j}(x, z)$ is an elementary landscape with $k = 2r$ when $z = r - 1$ and with $k = r$ when $z = -1$.

**Proof.** The function $\phi_{i,j}(x, z)$ returns the change in cost associated with edge $e_{i,j}$. Assume $\phi_{i,j}(x, z) = c$ when neither vertex $i$ nor vertex $j$ are recolored (i.e., the cost associated with edge $e_{i,j}$ does not change). If vertex $i$ or vertex $j$ is recolored, the value returned by $\phi_{i,j}(x, z)$ changes, even if it happens to result in a cost of the same value.

There are $d$ neighbors, but there are only $2(r-1)$ neighboring moves that recolor $i$ or $j$. Let $B(i,j)$ denote the average change in the cost associated with edge $e_{i,j}$ over the $2(r-1)$ cases where $i$ or $j$ are recolored and such that before recoloring $\phi_{i,j}(x, z) = c$. Then, the cumulative change over all neighbors is given by the formula:

$$d \cdot \text{avg}_{y \in N(x)} \{ \phi_{i,j}(y, z) \} = (d - 2(r-1))\phi_{i,j}(y, z) + 2(r-1)B(i,j)$$

and therefore by simple algebra the average over all neighbors is given by:

$$\text{avg}_{y \in N(x)} \{ \phi_{i,j}(y, z) \} = c + \frac{2(r-1)B(i,j) - 2(r-1)c}{d}$$

We use the previous expression to compute the average value in the neighborhood of the solutions. In the remainder of the proof we distinguish four different cases:

- **Case $\phi_{i,j}(x, z) = z$.** In this case the vertices $v_i$ and $v_j$ have different extreme frequencies. If we focus on the neighbor solutions we find that there are two neighbors with $\phi_{i,j}(y, z) = -1$ and $2(r-2)$ neighbors with $\phi_{i,j}(y, z) = 0$. It is not possible to obtain a neighbor with $\phi_{i,j}(y, z) = 1$.

  When this is simplified we find that the average value of $\phi_{i,j}(y, z)$ over the entire neighborhood is:

  $$\text{avg}_{y \in N(x)} \{ \phi_{i,j}(y, z) \} = z + \frac{-2 - 2(r-1)z}{d}$$

- **Case $\phi_{i,j}(x, z) = -1$.** In this case the vertices $v_i$ and $v_j$ have the same extreme frequency. There are two neighbors with $\phi_{i,j}(y, z) = z$ and $2(r-2)$ neighbors with $\phi_{i,j}(y, z) = 0$. It is not possible to obtain a neighbor with $\phi_{i,j}(y, z) = 1$. Then the average value of $\phi_{i,j}(y, z)$ in the neighborhood is:

  $$\text{avg}_{y \in N(x)} \{ \phi_{i,j}(y, z) \} = -1 + \frac{2z - 2(r-1)z}{d} = -1 + \frac{2(r + z - 1)}{d}$$

- **Case $\phi_{i,j}(x, z) = 1$.** In this case the vertices $v_i$ and $v_j$ have internal frequencies. There are four neighbors with $\phi_{i,j}(y, z) = 0$ and remaining neighbors have $\phi_{i,j}(y, z) = 1$. Then the average value of $\phi_{i,j}(y, z)$ in the neighborhood is:

  $$\text{avg}_{y \in N(x)} \{ \phi_{i,j}(y, z) \} = 1 + \frac{2(r-1) - 4 - 2(r-1)(1)}{d} = 1 - \frac{4}{d}$$

- **Case $\phi_{i,j}(x, z) = 0$.** In this case one vertex has an internal frequency and the other one has an extreme frequency. There is one neighbor with $\phi_{i,j}(y, z) = z$, one neighbor with $\phi_{i,j}(y, z) = -1$, $(r-2)$ neighbors with $\phi_{i,j}(y, z) = 1$, and the remaining neighbors have $\phi_{i,j}(y, z) = 0$. Then, the average value of $\phi_{i,j}(y, z)$ in the neighborhood is:

  $$\text{avg}_{y \in N(x)} \{ \phi_{i,j}(y, z) \} = 0 + \frac{z - 1 + (r-2)}{d} = \frac{r + z - 3}{d}$$

The function $\phi_{i,j}(y, z)$ is elementary if and only if there exist two constants $a$ and $b$ such that $\text{avg}_{y \in N(x)} \{ \phi_{i,j}(y, z) \} = a\phi_{i,j}(x, z) + b$. Now, we use the average values previously computed for the four possible cases and try to find the constants $a$ and $b$. To do this we need to solve the linear equation system:

$$
\begin{pmatrix}
    z & 1 & -1 & 1 & 0 \\
    -1 & 1 & 1 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
    a \\
    b
\end{pmatrix} =
\begin{pmatrix}
    z - 2 + 2(r-1)z \cdot d + (r + z - 3) \\
    -1 + 2(r - 1) \cdot d
\end{pmatrix}
$$

The previous system has four equations and two variables, so it could be unsolvable. However, the system can be solved when $z = r - 1$ and when $z = -1$, and the solution is $a = 1 - (r + z + 3)/d$ and $b = (r + z - 3)/d$. This means that we can write:

$$
\text{avg}_{y \in N(x)} \{ \phi_{i,j}(y, z) \} = \begin{cases}
    1 - \frac{r + z + 1}{d} + \frac{r + z - 3}{d} & \\
    \phi_{i,j}(x, z) + \frac{r + z + 1}{d} \cdot \phi_{i,j}(x, z) = \phi_{i,j}(x, z) + \frac{r + z + 1}{d} \cdot \phi_{i,j}(x, z)
\end{cases}
$$

where $\bar{\phi}_z$ denoted the average over the function. From the previous expression we conclude that all the $\phi_{i,j}(x, r - 1)$ functions are elementary landscapes with constant $k = 2r$ and average value $\bar{\phi}_{i,j} = (r - 2)/r$ and the $\phi_{i,j}(x, z)$ functions are elementary landscapes with constant $k = r$ and average value $\bar{\phi}_{i,j} = (r - 4)/r$. We also observe that:

$$
\bar{\phi}_z = \frac{r + z - 3}{r + z + 1} = \frac{2z - 2 + (r - 2)^2}{r^2}
$$

as we concluded from first principles. The last equality is valid only if $z = r - 1$ or $z = -1$.

If we substitute $\phi_{i,j}(x, r - 1)$ from $\phi_{i,j}(x, r - 1)$ we obtain the following result:

$$
\phi_{i,j}(x, r - 1) - \phi_{i,j}(x, -1) =
\begin{cases}
    r & \text{if } A(v_i, v_j) = \text{unequal extreme} \\
    0 & \text{if } A(v_i, v_j) = \text{equal extreme} \\
    0 & \text{if } A(v_i, v_j) = \text{internal} \\
    0 & \text{if } A(v_i, v_j) = \text{internal + extreme}
\end{cases}
$$

but this is $r \cdot \varphi_{i,j}(x)$, so we can write

$$
\varphi_{i,j}(x) = \frac{1}{r} (\phi_{i,j}(x, r - 1) - \phi_{i,j}(x, -1))
$$

(2)
We can now use two properties of elementary landscapes:

- The product of an elementary landscape by a scalar value is an elementary landscape with the same characteristic constant $k$.
- The sum of two elementary landscapes with the same characteristic constant $k$ is again an elementary landscape with the same $k$.

Using these two properties and the previous lemma we conclude that $\varphi_{i,j}(x)$ is the sum of two different elementary landscapes (with different characteristic constants). As a consequence, the function $f_{\bar{e},r}(x)$ is the sum of two different elementary landscapes, since it is the weighted sum of the different $\varphi_{i,j}(x)$ functions. And, finally, we get the following

**Theorem 2.** Using the standard “recoloring” neighborhood, the function $f_{aa}(x)$ is the superposition of two elementary landscapes with constants $k_1 = 2r$ and $k_2 = r$.

**Proof.** We have shown above that $f_{\bar{e},r}(x)$ is the sum of two elementary landscapes with constants $k_1 = 2r$ and $k_2 = r$. Let us write these two elementary landscapes as $f_{e,f,2r}$ and $f_{e,f,r}$, respectively. Their expressions are:

\[
\begin{align*}
f_{e,f,2r}(x) &= \sum_{e_{i,j} \in \mathcal{E}} \frac{m_i}{r} \phi_{i,j}(x, r - 1) \\
f_{e,f,r}(x) &= -\sum_{e_{i,j} \in \mathcal{E}} \frac{m_i}{r} \phi_{i,j}(x, -1)
\end{align*}
\]

Then, according to (1) we can write:

\[
f_{aa}(x) = f_{aa}(x) - f_{e,f,2r}(x) - f_{e,f,r}(x)
\]

The functions $f_{aa}$ and $f_{e,f,2r}$ are elementary landscapes with the same constant, $k_1 = 2r$, so their sum is also elementary with the same constant. Thus, we can express the $f_{aa}$ function as a sum of two elementary landscapes with constants $k_1 = 2r$ and $k_2 = r$:

\[
f_{aa}(x) = f_{aa,2r}(x) + f_{aa,r}(x)
\]

where

\[
\begin{align*}
f_{aa,2r}(x) &= f_{ba}(x) - f_{e,f,2r}(x) \\
f_{aa,r}(x) &= -f_{e,f,r}(x)
\end{align*}
\]

We mentioned above that one possible characterization of elementary landscapes is that there exist two constants $a$ and $b$ such that \( \text{avg}\{f(y)\}_{y \in \mathcal{N}(x)} = af(x) + b \). This means that there is a linear relationship between the average of the fitness function in the neighborhood of a solution $x$ and the fitness function of $x$ itself. When a fitness function is the superposition (sum) of two elementary landscapes, as it is the case in $f_{aa}$, this is not true. However, if one elementary component is known then we can compute the average of the fitness value in the neighborhood of a solution using a closed formula:

\[
\text{avg}\{f(y)\}_{y \in \mathcal{N}(x)} = \frac{f_1(x) + k_1}{d} (\bar{f}_1 - f_1(x)) + f_2(x) + \frac{k_2}{d} (f_2(x) - f_2(x)) = (1 - \frac{k_1}{d}) f(x) + \frac{k_1}{d} \bar{f}_1 + \frac{k_2}{d} \bar{f}_2
\]

![Figure 1: Elementary components $f_{aa,2r}(x)$ and $f_{aa,r}(x)$ against the fitness value $f_{aa}(x)$ for a set of random solutions.](image)

where $f_2$ is the known elementary component with constant $k_2$ and $k_1$ is the constant of the other elementary component. In the particular case of $f_{aa}$ we can write the previous expression as:

\[
\text{avg}\{f_{aa}(y)\}_{y \in \mathcal{N}(x)} = \left(1 - \frac{2r}{d}\right) f_{aa}(x) + \frac{r}{d} f_{aa,r}(x) + r \frac{f_{aa} + f_{aa,2r}}{d}
\]

(3)

In the rest of this section we present some figures to illustrate the concepts previously explained and to observe the relationships previously proven. The problem instance used is a real-world instance belonging to a mobile phone network having $|V| = 100$ base stations (nodes) and $r = 18$ different frequencies that can be assigned to them. This means a neighborhood size of $d = 1700$. The results shown in the figures correspond to 10,000 random solutions.

Let us first plot the value of the elementary components $f_{aa,2r}$ and $f_{aa,r}$ against the fitness value $f_{aa}$. The results are in Figure 1. We can observe that the values of $f_{aa,r}$ are around $10^6$ while the values of $f_{aa,2r}$ increase with the value of $f_{aa}$. We conclude that for high values of $f_{aa}$ the contribution of $f_{aa,r}$ to the fitness function is tiny.

In Figure 2 we plot the average value of the fitness function in the neighborhood of a solution, $\text{avg}\{f_{aa}(y)\}_{y \in \mathcal{N}(x)}$, against the fitness function of the solution itself, $f_{aa}(x)$. We can observe that the relationship between these two parameters is almost linear with slope of approximately 1. This plot can be easily explained using Equation (3): the average in the neighborhood is the sum of two varying terms and a constant; $f_{aa}(x)$ is multiplied by $1 - 2r/d$ and $f_{aa,r}$ is multiplied by $r/d$. In real-world instances of this problem, the number of frequencies, $r$, is much smaller than the size of the neighborhood, $d$, since the number of nodes in the graph, $|V|$, is large. This means that the multiplier of $f_{aa}$ is approximately 1 and the multiplier of $f_{aa,r}$ is very small. In this particular instance of the problem, the multiplier of $f_{aa}$ is $1 - 36/1700 \approx 0.9788$; and the multiplier of $f_{aa,r}$ is $18/1700 \approx 0.0106$. From Figure 1 we conclude that $f_{aa,r}(x) \approx 10^6$ and when we multiply this value by 0.0106 we obtain a value around $10^4$ which is negligible with respect to 0.9788 $f_{aa} \gtrsim 10^6$. This is the reason why a straight line appears in Figure 2. Even although $f_{aa}$ is not an elemen-
min of the function in the neighborhood.

By the elementary landscape, for this instance it behaves almost like an elementary landscape.

Finally, in Figure 3 we plot the minimum fitness value of a neighbor of \( x \), \( \min_{y \in N(x)} f_{aa}(y) \) against the fitness value of \( x \), \( f_{aa}(x) \). We can observe a linear trend. This is an interesting and important feature. We know that in elementary landscapes there is a linear relationship between the fitness value of a solution and the average fitness value in their neighborhood. Thus, if \( f(x) < f(x') \) then \( \text{avg}\{f(y)\}_{y \in N(x)} < \text{avg}\{f(y')\}_{y' \in N(x')} \). But this does not ensure that we can find a solution in \( N(x) \) with a fitness value that is lower than the one of all solutions in \( N(x') \). As we argue in the previous paragraph, \( f_{aa} \) is almost elementary, so we can apply the argument. The linear trend shown in Figure 3 means that the lower the value of \( f_{aa} \), the lower the minimum value of the function in the neighborhood.

3.4 Partial Neighborhoods

Whitley and Sutton [11] calculated the average of a partial neighborhood for the Graph Coloring problem where the moves are restricted to those that remove conflicts that contribute to the cost function. We present a new theorem that generalizes their results so as to hold for the Basic FAP (Weighted Graph Coloring) and for a larger class of partial neighborhoods. We also correct an omission in their proof and show the theorem they present is correct.

We define a vector \( S \) in terms of component of matrix \( M \).

\[
S_i = \sum_{j=1}^{|V|} (m_{ij} + m_{ji}) \quad \forall i = 1, 2, \ldots, |V|
\]

Since \( M \) is in lower triangle form, either \( m_{ij} \) or \( m_{ji} \) will be zero. Each element \( S_i \) in vector \( S \) is the sum of all the costs in the matrix \( M \) associated with edges that are incident on vertex \( v_i \). This means that every entry \( m_{jk} \) in the cost matrix contributes to both \( S_i \) and to \( S_j \) in \( S \). Therefore if we sum over the components of \( S \) it follows that:

\[
\sum_{i=1}^{|V|} S_i = 2 \sum_{v \in C} \sum_{j=1}^{|V|} S_j \sum_{k=1}^{|V|} m_{jk}
\]

Thus the vector \( S \) sums over the set of edges twice. Let \( Q_X \) be the set of vertices such that if \( m_{ij} \) contributes cost to \( f(x) \) then vertices \( i \) and \( j \) are members of set \( Q_X \). Let \( Q_x \) be any subset of \( Q_X \). Note this is more general than the Whitley and Sutton theorem.

**Theorem 3.** Let \( N'(x) \) be a partial neighborhood such that only vertices in \( Q_x \) are recolored; the neighborhood average for partial neighborhood \( Q_x \) for the Weighted Graph Coloring problem is given by:

\[
\text{avg}\{f(y)\} = f(x) + \sum_{v \in Q_x} S_i - 2rf(x) \frac{|Q_x|}{|Q_x|}
\]

**Proof.** Only vertices in \( Q_x \) will be recolored, and the size of the neighborhood changes to \( |Q_x| \). But the number of ways a vertex can remove conflicts \( (\alpha) \) does not change, therefore:

\[
p_1 = \frac{2(r-1)}{|Q_x|}
\]

The edges that contribute to \( f(x) \) can be separated into three groups.

1. Edges that do not contribute to \( f(x) \) and are not incident on a vertex in \( Q_x \).
2. Edges that do not contribute to \( f(x) \) and are incident on one vertex in \( Q_x \).
3. Edges that do not contribute to \( f(x) \) but which are incident on two vertices in \( Q_x \). Whitley and Sutton’s proof failed to cover this case.

The costs associated with all of the vertices that are not being recolored are computed using elements of the vector \( S \):

\[
\sum_{i \in v \in V - Q_x} S_i
\]

To see that this summation is correct, consider vertices \( v_j \) and \( v_k \). If neither vertex is recolored, then \( m_{jk} \) is counted once. If only one of the vertices is recolored, then \( m_{jk} \) is counted twice. If both of these vertices appear in \( Q_x \) then \( m_{jk} \) is not counted.

Note \( \sum_{i=1}^{|V|} S_i \) counts all costs twice and \( \sum_{i=1}^{|V|} S_i - 2f(x) \) removes the costs associated with \( f(x) \) twice, and the remaining pairs of vertices have the associated cost removed if neither vertex is recolored, and the cost is included once.
if only one vertex is recolored; the cost is included twice if both vertices are recolored. Therefore, the set of edges in graph $G$ that do not contribute to $f(x)$ and which change when the vertices in $Q_x$ are recolored is given by

$$\sum_{i=1}^{\left|V\right|} S_i - 2f(x) - \sum_{i|v_i \in V - Q_x} S_i = \sum_{i|v_i \in Q_x} S_i - 2f(x)$$

Consider an edge where one vertex is in $Q_x$ and one vertex is not. There is only 1 way to generate a conflict, and $\beta = 1$.

$$p_2 = \frac{1}{|Q_x|(r - 1)}$$

Consider an edge where both vertices are in $Q_x$; then in this case, we still use $\beta = 1$ but the contribution of this edge to the quantity $\sum_{i|v_i \in Q_x} S_i - 2f(x)$ has already been counted twice to correct for this difference. We now apply the component model to the partial neighborhood.

$$\text{avg}(f(y)) = f(x) - \frac{2(r-1)}{2} f(x) + \frac{1}{|Q_x|} \sum_{i|v_i \in Q_x} S_i - 2f(x)$$

$$= f(x) + \frac{\sum_{i|v_i \in Q_x} S_i - 2f(x) - 2(r-1) f(x)}{|Q_x|(r - 1)}$$

$$= f(x) + \frac{\sum_{i|v_i \in Q_x} S_i - 2r f(x)}{|Q_x|(r - 1)}$$

4. CONCLUSIONS

In this paper we have analyzed three variants of the Frequency Assignment Problem from the point of view of the elementary landscape theory. The first variant, called Basic FAP, is a Weighted Graph Coloring Problem. This problem is an elementary landscape. The second variant, called Symmetric Adjacent FAP, considers only conflicts between adjacent frequencies and assumes that the lowest and the highest frequencies are a source of conflicts. This variant is also elementary. The third variant, called Asymmetric Adjacent FAP, takes into account direct conflicts and adjacent conflicts without symmetry. This variant is the superposition of two elementary landscapes with different characteristic constants.

It is reasonable to ask what we have gained by establishing that the basic Frequency Assignment Problem and Symmetric Adjacent FAP both have an elementary landscape (under an appropriate operator) and that the Asymmetric Adjacent FAP is a superposition of two elementary landscapes. By establishing that variants of the Frequency Assignment Problem correspond to elementary landscapes, we lay the foundation for leveraging other results that hold for other elementary landscapes. It has been shown [10] that the exact autocorrelation of the search landscape can be computed in closed form in polynomial time for all elementary landscapes which are $k$-bounded pseudo-boolean functions; this includes MAX-SAT and NK-Landscapes. While FAP is not a pseudo-boolean function, we nevertheless have recent proven this result can be extended to the FAP. Whitley et al. [12] also developed a “tunneling algorithm” for the TSP that is directly able to construct new local optima by decomposing and reconfiguring known local optima. This algorithm exploits the fact that the fitness function is a linear combination of “components.” We have also recent proven that a similar “tunneling algorithm” can be constructed for variants of the Frequency Assignment Problem. This could open the door to much more efficient forms of search.

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6. REFERENCES


