CS320 First Midterm Exam
Fall, 2018, Section 002
100 total points

1. (5 points) Which of these complexity measures would you use to pick the algorithm that will complete the quickest for a large value of \( n \)? Circle all correct answers.

   \( O \)  \( \Omega \)  \( \Theta \)

2. (5 points) Which of these complexity measures would you use to get the most precise estimate of the running time of an algorithm for a given value of \( n \)? Circle all correct answers.

   \( O \)  \( \Omega \)  \( \Theta \)

3. (5 points) Show that an algorithm that requires \( n^2 + n/2 \) steps is \( O(n^2) \).

   \[ n^2 + \frac{n}{2} \leq Cn^2 \quad \text{for } n \geq n_0 \]

   \[ \Theta(n + \frac{1}{2}) \leq Cn^2 \]

   \[ 1 + \frac{1}{2n} \leq C \quad n \geq 1 \]

   \[ 1 + \frac{1}{2000} \leq C \quad n \geq 1000 \]

   \[ C = 1.001 \]

4. (10 points) Show that an algorithm that requires \( n \log n \) steps is \( \Omega(\sqrt{n}) \).

   \[ C \sqrt{n} \leq n \log n \quad \text{for } n \geq n_0 \]

   \[ C \leq \sqrt{n} \log n \quad n \geq 42 \]

   \[ C = 1 \]
5. (10 points) Circle which of the following are true. At least one is, but both may be true. Justify your answer.

\[ a^n = O(n!) \quad n! = O(a^n) \quad \text{for } a > 0 \]

\[ a^n \leq c \cdot n! \quad n \geq n_0 \]

\[ a \leq c \cdot n(n-1)(n-2) \cdots 1 \]

\[ n(n-1)(n-2) \cdots k(k-1) \cdots 1 \]

\[ \geq a^{n-k} \leq a^k \]

\[ n \log a \leq \log(c) + \log n + \log(n-1) + \cdots + \log 1 \]

\[ \log a \leq \frac{n}{n} \log(c) + \frac{1}{n} \log n \]

6. (10 points) Show that \( \frac{n \log n}{\log_2 n} = \Omega(n \log_8 n) \).

\[ \log x = \frac{\log_a x}{\log_a b} \]

\[ \log a x = \log b x \log a b \]

\[ \log a x = \log b x \log a b \]

7. (5 points) Can an algorithm that is \( \Omega(n) \) be \( \Theta(n^2) \)? Why or why not?

\[ f(n) = \log_2^2 n \]

\[ f(n) = \log_3 (\log n) \]

\[ f(n) = \Omega(n) \]

\[ f(n) = \Theta(n^2) \]

\[ f(n) = \Theta(n) \]

\[ f(n) = O(n^4) \]
8. (5 points) An implementation of a max heap that uses an array of values maintains those values in the array in descending order. Circle the correct answer.

True  False

9. (5 points) How many edges does a tree with \( n \) nodes have?

\[ n - 1 \]

10. (5 points) Does this binary tree satisfy the max heap property?

YES

11. (5 points) Draw the max heap that results when the value 31 is added to the above max heap.
12. (5 points) Draw a graph that is not bipartite.

13. (5 points) A directed acyclic graph can have more than one topological ordering of its vertices. Circle the correct answer.

True
False

14. (10 points) Use Prim's algorithm to find a minimum spanning tree for the following graph. Shade in the vertex which you choose as the starting vertex for Prim’s algorithm. Thicken each edge that you include in the minimum spanning tree. Label the edges with a, b, c, ... as you add them to the minimum spanning tree.
15. (10 points) Prove the following equation using induction. Clearly show the base case, the inductive hypothesis, and the inductive step.

\[ \sum_{i=0}^{n} 2^i = 2^{n+1} - 1, \text{ for } n \geq 0 \]

**Base Case** \((n=0)\)

\[ \sum_{i=0}^{0} 2^i = 2^0 - 1 = 1 \]

**Ind. Hyp.** \((n=k)\)

\[ \sum_{i=0}^{k} 2^i = 2^{k+1} - 1 \]

**Inductive Step** If true for \(n=k\), show true for \(n=k+1\)

\[ 2^{k+1} + \sum_{i=0}^{k} 2^i = 2^{k+2} - 1 \]

\[ 2^{k+1} + 2 - 1 = 2^{k+2} - 1 \]