Permutations

A permutation of a set of distinct objects is an ordered arrangement of these objects. Example: (1, 3, 2, 4) is a permutation of the numbers 1, 2, 3, 4.

How many permutations of \( n \) objects are there?

Using the product rule:
\[
\begin{align*}
\frac{n}{1} \cdot \frac{n-1}{2} \cdot \ldots \cdot \frac{n-2}{2} \cdot \frac{2}{1} = n!
\end{align*}
\]

The Traveling Salesman Problem (TSP)

TSP: Given a list of cities and their pairwise distances, find a shortest possible tour that visits each city exactly once.

Objective: find a permutation \(a_1, a_2, \ldots, a_n\) of the cities that minimizes
\[
\begin{align*}
d(a_1, a_2) + d(a_2, a_3) + \ldots + d(a_{n-1}, a_n) + d(a_n, a_1)
\end{align*}
\]

where \(d_{ij}\) is the distance between cities \(i\) and \(j\).
Generating Permutations

- **Lexicographic** (dictionary) ordering: A permutation \(a_1a_2\ldots a_n\) precedes \(b_1b_2\ldots b_n\) if for some \(k\) \(a_k = b_1 = \ldots = a_{k-1} = b_{k-1}\), and \(a_k < b_k\).
- Example: The permutation 23415 precedes 23514

Generating Permutations: Algorithm

- Find the next largest permutation after \(a_1a_2\ldots a_n\)
  - Find the largest position \(i\) such that \(a_i < a_{i+1}\).
  - Replace \(a_i\) with the smallest value in \(a_{i+1}\ldots a_n\) that is bigger than the current value of \(a_i\).
  - Sort the remaining elements \(a_{i+1}\ldots a_n\) (and the original value of \(a_i\)) in increasing order.
- Apply the algorithm to 812369754

\(r\)-permutations

- An ordered arrangement of \(r\) elements of a set: \(r\)-permutations
- The number of \(r\)-permutations of a set with elements: \(P(n,r)\)
- Example: List the \(r\)-permutations of \((a,b,c)\).
  \(P(3,2) = 3 \times 2 = 6\)
- Let \(n\) and \(r\) be integers such that \(0 \leq r \leq n\) then there are \(P(n,r) = n(n-1)\ldots(n-r+1)\)
- \(r\)-permutations of a set with \(n\) elements.
  \(P(n,r) = n! / (n-r)!\)

\(r\)-permutations - example

- How many ways are there to select a first-prize winner, a second prize winner and a third prize winner from 100 people who have entered a contest?

Combinations

- How many poker hands (five cards) can be dealt from a deck of 52 cards?
- How is this different than \(r\)-permutations?
Combinations

- An \( r \)-combination of a set is a subset with \( r \) elements.
- Example: \( \{1,3,4\} \) is a 3-combination of \( \{1,2,3,4\} \).
- The number of \( r \)-combinations out of a set with \( n \) elements: \( C(n,r) \) also denoted as: \( \binom{n}{r} \).
- Example: How many 2-combinations of \( \{a,b,c,d\} \).

\[ C(n, r) = \frac{n!}{r!(n-r)!} \]

Notice: \( C(n, r) = C(n, n - r) \)

Can prove that without using the formula.

Unordered versus ordered selections

- Two ordered selections are the same if
  - the elements chosen are the same;
  - the elements chosen are in the same order.
- Ordered selections: \( r \)-permutations.
- Two unordered selections are the same if
  - the elements chosen are the same.
    (regardless of the order in which the elements are chosen)
- Unordered selections: \( r \)-combinations.

Relationship between \( P(n,r) \) and \( C(n,r) \)

- Suppose we want to compute \( P(n,r) \).
- Constructing an \( r \)-permutation from a set of \( n \) elements can be thought as a 2-step process:
  - Step 1: Choose a subset of \( r \) elements;
  - Step 2: Choose an ordering of the \( r \)-element subset.
- Step 1 can be done in \( C(n,r) \) different ways.
- Step 2 can be done in \( r! \) different ways (regardless of how the step 1 was performed).
- Based on the multiplication rule, \( P(n,r) = C(n,r) \cdot r! \).
- Thus,
  \[ C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!} \]

r-combinations

- Example: How many bit strings of length \( n \) contain exactly \( r \) ones?

More Examples

- How many subsets with more than two elements does a set with 100 elements have?
- Seven women and nine men are on the faculty in a CS dept. How many ways are there to select a committee of 4, if at least one woman must be on the committee?
Example

4 members from a group of 11 are supposed to work as a team on a project.
Q: How many distinct 4-person teams can be chosen?

Suppose, Mary will work on the project only if John is also involved in the project.
Q: What is the number of 4-people teams that can be chosen from 11 people in this case?
(hint: divide the set of all possible teams into two disjoint subsets)

The Difference Rule

- If A is a finite set and B is a subset of A, then \(|A - B| = |A| - |B|\).
- Example: Assume that any seven digits can be used to form a telephone number.
  Q: How many seven-digit phone numbers have at least one repeated digit?
  Let \(A\) = the set of all possible 7-digit phone numbers;
  \(B\) = the set of 7-digit numbers without repetition.
  Note that \(B \subseteq A\).
  Then \(A - B\) is the set of 7-digit numbers with repetition.
  \[|A - B| = |A| - |B| \quad \text{(by the difference rule)}\]
  \[= 10^7 - P(10,7) \quad \text{(7-permutation)}\]
  \[= 10^7 - 10!/3!\]

Examples on Combinations

- Suppose that 3 cars in a production run of 40 are defective. A sample of 4 is to be selected to be checked for defects.
- Questions:
  1) How many different samples can be chosen?
  2) How many samples will contain exactly one defective car?
  3) How many samples will contain at least one defective car?

Some Advice about Counting

- Apply the multiplication rule if
  - The elements to be counted can be obtained through a multistep selection process;
  - Each step is performed in a fixed number of ways regardless of how preceding steps were performed.
- Apply the addition rule if
  - The set of elements to be counted can be broken up into disjoint subsets.
- Note: Often a counting problem is solved by applying both the multiplication and addition rules (and their variations) at different stages of the solution.
Some Advice about Counting

- Make sure that
  1) every element is counted;
  2) no element is counted more than once.
  (avoid double counting)
- When using the addition rule:
  1) every outcome should be in some subset;
  2) the subsets should be disjoint.

Example using Inclusion/Exclusion Rule

- Question: How many integers from 1 through 100 are multiples of 4 or multiples of 6?
- Solution: Let A be the set of integers from 1 through 100 which are multiples of 4; B be the set of integers from 1 through 100 which are multiples of 6. We want to find $|A \cup B|$.
- $A \cap B$ is the set of integers from 1 through 100 which are multiples of 12.
- $|A \cup B| = |A| + |B| - |A \cap B|$ (by incl./excl. rule)
  $= 25 + 16 - 8 = 33$ (by counting)