CS 200: Relations

Rosen 8.1 – 8.5

Tuples

- An ordered n-tuple is a sequence of n objects
  \((x_1, x_2, \ldots, x_n)\)
  First component is \(x_1\)
  …
  n-th component is \(x_n\)
- An ordered pair: 2-tuple \((x, y)\)
- An ordered triple: 3-tuple \((x, y, z)\)

Tuples vs Sets

- Two tuples are equal iff they are equal coordinate-wise
  \((x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n)\) iff
  \(x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n\)

  \((2, 1) \neq (1, 2), \text{ but } \{2, 1\} = \{1, 2\}\)
  \((1, 2, 1) \neq (2, 1), \text{ but } \{1, 2, 1\} = \{2, 1\}\)

Binary Relations

- A – set of students  B – set of courses
  R – pairs \((a, b)\) such that student a is enrolled in course b
  R = \{\((\text{chris, cs200}), (\text{mike, cs520}), \ldots\)\}
- A – set of cities  B – set of US states
  R – \((a, b)\) such that city a is in state b
  R = \{\((\text{Denver, CO}), (\text{Laramie, WY}), \ldots\)\}

Binary Relations

- A binary relation from a set A to a set B is a set R of ordered pairs \((a, b)\) where \(a \in A\) and \(b \in B\).
- The notation \(aRb\) denotes \((a, b) \in R\)
- Example: \(A = \{0, 1, 2\}, B = \{a, b\}\) and \(R = \{(0, a), (0, b), (1, a), (2, b)\}\)

Relations as Cartesian Products

- Let A, B be sets
  The \textbf{cartesian product} of A and B is denoted by \(A \times B\) and is equal to:
  \[ \{(a, b) \mid a \in A \text{ and } b \in B\} \]
- A binary relation from A to B is a subset of \(A \times B\)
- Given sets A and B with sizes n and m the number of elements in \(A \times B\) is nm and the number of binary relations from A to B is \(2^{nm}\)
**n-ary Relations**

**Definition:** Let \( A_1, A_2, \ldots, A_n \) be sets. An *n-ary relation* on these sets is a subset of \( A_1 \times A_2 \times \cdots \times A_n \).

The sets \( A_1, A_2, \ldots, A_n \) are called the *domains* of the relation, and \( n \) is called its *degree*.

**Example:** The *between* relation consisting of triples \((a,b,c)\) where \(a, b, c\) are integers such that \(a < b < c\).

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**Databases and Relations**

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</table>

Databases defined by relations are called *relational databases*.

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**Functions as Relations**

- A function \( f \) from \( A \) to \( B \) assigns an element of \( B \) to each element of \( A \).
- Differences between relations and functions?

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**Relations on a Set**

- A relation on a set \( A \) is a relation from \( A \) to \( A \).
- Example: relations on the set of integers
  - \( R_1 = \{(a,b) \mid a \leq b\} \)
  - \( R_2 = \{(a,b) \mid a > b\} \)
  - \( R_3 = \{(a,b) \mid a = b + 1\} \)

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**Relations on a Set as Graphs**

- Consider the relation \( R \) on cities:
  \[ R = \{(a,b) \mid a, b \text{ are cities such that the population of } a \text{ is smaller than that of } b\} \]
- We can represent \( R \) as a directed graph where there is an edge from \( a \) to \( b \) if \((a,b)\) is in \( R \).
Relations on a Set

- A – the set of actors
- \( R = \{(a, b) : a, b \text{ are actors that have played in the same movie}\) 

R has the property that if \( aRb \) then \( bRa \).
- It is a symmetric relation
- Can be represented as an undirected graph

Properties of Relations

- A relation \( R \) on a set \( A \) is called transitive if whenever \( aRb \) and \( bRc \) then \( aRc \) for all \( a, b, c \) in \( A \).
- Example: the ancestor relation

- A relation \( R \) on a set \( A \) is called reflexive if \( aRa \) for all \( a \) in \( A \).
- Example: the less-or-equal to relation on the positive integers

Composite Relations

- Let \( R \) be a relation from \( A \) to \( B \), and let \( S \) be a relation from \( B \) to \( C \). The composite \( S \circ R \) of \( R \) and \( S \) is defined as:
  \[ S \circ R = \{(a, c) \mid \exists b : aRb \land bSc\} \]
- Example: Let \( R \) be the relation such that \( aRb \) if \( a \) is a parent of \( b \). What is the relation \( R \circ R \)?

Composite Relations

- \( R^2 = R \circ R = (a, c) \quad (e, c) \quad (b, d) \)
  \[ (d, d) \quad (c, c) \]

Paths and Relations

- Let \( R \) be a relation on a set \( A \). There is a path of length \( n \) from \( a \) to \( b \) in the graph representing \( R \) if and only if \( aR^n b \)
- Example: the six-degrees of separation can be succinctly expressed as \( aR^6 b \) where \( R \) is the relation on the set of people such that \( aRb \) if \( a \) knows \( b \)
- Proof: by induction on \( n \) (Rosen p. 547)
Paths and Relations

- Define: $R^* = \bigcup_{i=1}^{\infty} R^i$

  - Example: Let $R$ be the relation between states in the US where $aRb$ if $a$ and $b$ share a common border. What is $R^*$?

  - For a relation over a set with $n$ elements $\quad R^* = \bigcup_{i=1}^{n} R^i$

Transitive Closure

- The smallest transitive relation on $A$ that includes $R$ is called the transitive closure of $R$.

  - $R^*$ is the transitive closure of $R$ (Rosen p. 548)

  - Example: $A = \{1, 2, b\}$
    - $R = \{(1, 1), (b, b)\}$
    - $S = \{(1, 2), (2, b), (1, b)\}$
    - $T = \{(2, b), (b, 2), (1, 1)\}$

  - The transitive closures of $R$ and $S$ are themselves

  - The transitive closure of $T$ is $T \cup \{(2, 2), (b, b)\}$