Uncertainty

Let $A_t$ be the action of leaving for the airport $t$ minutes before flight. Will $A_t$ get you there on time?

Uncertainty as a result of:
1. partial observability (road state, other drivers' plans, etc.)
2. noisy sensors (traffic reports)
3. uncertainty in action outcomes (flat tire, etc.)
4. complexity of modeling traffic

Questions

- How to represent uncertainty in knowledge?
- How to perform inference with uncertain knowledge?
- Which action to choose under uncertainty?

Dealing with uncertainty

- Implicit
  - Ignore what you are uncertain of when you can
  - Build procedures that are robust to uncertainty

- Explicit
  - Build a model of the world that describes uncertainty about its state, dynamics, and observations
  - Reason about the effect of actions given the model

Methods for handling uncertainty

- Default Reasoning:
  - Assume the car does not have a flat tire
  - Assume $A_{120}$ works unless contradicted by evidence
  - Issues: What assumptions are reasonable? How to handle contradictions?

- Worst case reasoning (the world behaves according to Murphy’s law).

- Probability
  - Model agent’s degree of belief
  - Given the available evidence, $A_{120}$ will get me there on time with probability 0.95
Probability

- Probabilities relate propositions to agent's own state of knowledge
  - e.g., \( P(A_{120} \mid \text{no reported accidents}) = 0.96 \)
- Probabilities of propositions change with new evidence:
  - e.g., \( P(A_{120} \mid \text{no reported accidents, 5 a.m.}) = 0.99 \)

Making decisions under uncertainty

Suppose I believe the following:

\[
\begin{align*}
P(A_{25} \text{ gets me there on time | ...}) &= 0.001 \\
P(A_{90} \text{ gets me there on time | ...}) &= 0.70 \\
P(A_{120} \text{ gets me there on time | ...}) &= 0.95 \\
P(A_{1440} \text{ gets me there on time | ...}) &= 0.9999
\end{align*}
\]

- Which action to choose?
  - Depends on my preferences for missing flight vs. time spent waiting, etc.
  - Utility theory is used to represent and infer preferences
  - Decision theory = probability theory + utility theory

Axioms of probability

- For any events \( A, B \) in a space of events \( \Omega \)
  - \( 0 \leq P(A) \leq 1 \)
  - \( P(\Omega) = 1 \) and \( P(\emptyset) = 0 \)
  - \( P(A \cup B) = P(A) + P(B) - P(A \cap B) \)

Example

- Consider a deck of cards (52 cards) and the following events:
  - A king
  - A face card
  - A spade
  - A face card or a red suit
  - A card
- What is the probability of each of the above events?

Where do probabilities come from

Two camps:
- Frequentist interpretation
- Bayesian interpretation

Frequentist interpretation

- Draw a ball from a urn containing \( n \) balls of the same size, \( r \) red the rest black.
- The probability of the event “the ball is red” corresponds to the relative frequency with which we expect to draw a red ball
  \[ P(\text{red}) = ? \]
Subjective probabilities

There are many situations in which there is no objective frequency interpretation:

- You have worked hard on your AI class and you believe that the probability that you will get an A is 0.9
- There are theoretical justifications for subjective probabilities!

The Bayesian viewpoint

- Probability is “degree-of-belief”.
- To the Bayesian, probability lies subjectively in the mind, and can be different for people with different information
- In contrast, to the frequentist, probability lies objectively in the external world.

Random Variables

- A random variable can be thought of as an unknown value that may change every time it is inspected.
- Suppose that a coin is tossed three times and the sequence of heads and tails is noted. The sample space for this experiment is:
  \[ S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\} \]
- \( X \) - the number of heads in three coin tosses. \( X \) assigns each outcome in \( S \) a number from the set \( \{0, 1, 2, 3\} \).

Random Variables

- **Boolean** random variables
  - e.g., Cavity (do I have a cavity?)
  - Distribution characterized by a number \( p \).
- **Discrete** random variables
  - e.g., Weather is one of <sunny,rainy,cloudy,snow>
  - Domain values must be exhaustive and mutually exclusive
  - The (probability) distribution of a random variable \( X \) with \( m \) values \( x_1, x_2, \ldots, x_m \): \[ (p_1, p_2, \ldots, p_m) \]
  - with \( P(X = x_i) = p_i \) and \( \sum p_i = 1 \)

Joint Distribution

- Given \( n \) random variables \( X_1, \ldots, X_n \)
- The joint distribution of these variables is a table in which each entry gives the probability of one combination of values of \( X_1, \ldots, X_n \)
- Example:

<table>
<thead>
<tr>
<th></th>
<th>Toothache</th>
<th>¬Toothache</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cavity</td>
<td>0.04</td>
<td>0.06</td>
</tr>
<tr>
<td>¬Cavity</td>
<td>0.01</td>
<td>0.89</td>
</tr>
</tbody>
</table>

\[ P(\neg\text{Cavity} \& \text{Toothache}) \quad P(\text{Cavity} \& \neg\text{Toothache}) \]

It’s all in the joint

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- \( P(\text{Toothache}) = P(\text{Toothache} \& \text{Cavity}) + P(\text{Toothache} \& \neg\text{Cavity}) \)
- \[ = 0.04 + 0.01 = 0.05 \]

We summed over all values of Cavity: **marginalization**

- \( P(\text{Toothache} \& \text{Cavity}) = P(\text{Toothache} \& \text{Cavity}) \quad \neg(\text{Toothache} \& \text{Cavity}) \)
- \[ = 0.04 + 0.01 + 0.06 + 0.11 \]

These are examples of inference by enumeration
Conditional Probability

- Definition: $P(A|B) = \frac{P(A \land B)}{P(B)} \quad (\text{if } P(B) > 0)$
- Read: probability of A given B
- Example: $P(\text{snow}) = 0.03$ but $P(\text{snow} \mid \text{winter}) = 0.06$, $P(\text{snow} \mid \text{summer}) = 1e-6$
- can also write this as: $P(A \land B) = P(A|B) \cdot P(B)$
- called the product rule

Example

<table>
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$P(\text{Cavity} \mid \text{Toothache}) = \frac{P(\text{Cavity} \land \text{Toothache})}{P(\text{Toothache})} = \frac{0.04}{0.05} = 0.8$

Product rule

- $P(A \land B \land C) = P(A \mid B, C) \cdot P(B \mid C) \cdot P(C)$
- $P(A \land B) = P(A \mid B) \cdot P(B)$
- $P(B \mid A) = \frac{P(A \mid B) \cdot P(B)}{P(A)}$

Bayes’ Rule

The Monty Hall Problem

Suppose you’re on a game show, and you’re given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what’s behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

Given:
- $P(\text{Cavity}) = 0.1$
- $P(\text{Toothache}) = 0.05$
- $P(\text{Cavity} \mid \text{Toothache}) = 0.8$

Using Bayes’ rule:
- $P(\text{Toothache} \mid \text{Cavity}) = \frac{(0.8 \times 0.05)}{0.1} = 0.4$

source: http://en.wikipedia.org/wiki/Monty_Hall_problem
Solution

\[ P(C_2|O_3) = \frac{P(O_3|C_2)P(C_2)}{P(O_3)} = \frac{1 \times 1/3}{1/2} = 2/3 \]

\[ P(O_3) = P(O_3|C_1)P(C_1) + P(O_3|C_2)P(C_2) + P(O_3|C_3)P(C_3) = \]
\[ = \frac{1/2 \times 1}{3} + \frac{1 \times 1}{3} + 0 \times \frac{1}{3} = \frac{1}{2} \]

Generalization of Bayes’ rule

\[ P(A \land B \land C) = P(A \land B|C) P(C) \]
\[ = P(A|B,C) P(B|C) P(C) \]

\[ P(B|A,C) = \frac{P(A|B,C) P(B|C)}{P(A|C)} \]

It’s all in the joint but…

- The naïve representation runs into problems.
- Example:
  - Patients in a hospital are described by attributes such as:
    - Background: age, gender, history of diseases, ...
    - Symptoms: fever, blood pressure, headache, ...
    - Diseases: pneumonia, heart attack, ...
- A probability distribution needs to assign a number to each combination of values of these attributes
  - Size of table is exponential in number of attributes

Bayesian networks

- Provide an efficient representation that relies on independence relations between variables.