



Proof Techniques (Rosen, Sections 1.5, 1.6, 1.7)

TOPICS

- Direct Proofs
- Proof by Contrapositive
- Proof by Contradiction
- Proof by Cases



Proof Terminology

Theorem: statement that can be shown to be true

Proof: a valid argument that establishes the truth of a theorem

Axioms: statements we assume to be true

Lemma: a less important theorem that is helpful in the proof of other results

Corollary: theorem that can be established directly from a theorem that has been proved

Conjecture: statement that is being *proposed* to be a true statement



Learning objectives

- Direct proofs
- Proof by contrapositive
- Proof by contradiction
- Proof by cases



Technique #1: Direct Proof

- Direct Proof:
 - First step is to clearly state the premise
 - Subsequent steps use rules of inference or other premises
 - Last step proves the conclusion



Direct Proof Example

- Prove “If n is an odd integer, then n^2 is odd.”

– If n is odd, then $n = 2k+1$ for some integer k .

– $n^2 = (2k+1)^2 = 4k^2 + 4k + 1$

– Therefore, $n^2 = 2(2k^2 + 2k) + 1$, which is odd.

2 * any # \rightarrow even

Add 1 to any even # \rightarrow odd #



More formal version...

Step	Reason
1. n is odd	Premise
2. $\exists k \in \mathbb{Z} \ n = 2k+1$	Def of odd integer in (1)
3. $n^2 = (2k+1)^2$	Squaring (2)
4. $= 4k^2 + 4k + 1$	Algebra on (3)
5. $= 2(2k^2 + 2k) + 1$	Algebra on (4)
6. $\therefore n^2$ is odd	Def odd int, from (5)



Class Exercise

- Prove: If n is an even integer, then n^2 is even.

– If n is even, then $n = 2k$ for some integer k .

– $n^2 = (2k)^2 = 4k^2$

– Therefore, $n = 2(2k^2)$, which is even.



Can you do the formal version?

Step	Reason
1. n is even	Premise
2. $\exists k \in \mathbb{Z} \ n = 2k$	Def of even integer in (1)
3. $n^2 = (2k)^2$	Squaring (2)
4. $= 4k^2$	Algebra on (3)
5. $= 2(2k^2)$	Algebra on (4)
6. $\therefore n^2$ is even	Def even int, from (5)



Technique #2: Proof by Contrapositive

- A direct proof, but starting with the contrapositive equivalence:
 $p \rightarrow q \equiv \neg q \rightarrow \neg p$
- If you are asked to prove $p \rightarrow q$, ...
- ..., you instead prove $\neg q \rightarrow \neg p$!
- Why? Sometimes, it may be easier to directly prove $\neg q \rightarrow \neg p$ than $p \rightarrow q$



Proof by Contrapositive Example

- Prove: If n^2 is an even integer, then n is even.
 $(n^2 \text{ even}) \rightarrow (n \text{ even})$
- By the contrapositive: This is the same as showing that
 $\neg(n \text{ even}) \rightarrow \neg(n^2 \text{ even})$
- If n is odd, then n^2 is odd. (proved on slides 4 and 5)
- Since we have proved the contrapositive:
 $\neg(n \text{ even}) \rightarrow \neg(n^2 \text{ even})$
- We have also proved the original hypothesis:
 $(n^2 \text{ even}) \rightarrow (n \text{ even})$



Technique #3: Proof by Contradiction

Prove: If p then q .

Proof strategy:

- Assume p and the negation of q .
- In other words, assume that $p \wedge \neg q$ is true.
- Then arrive at a contradiction $p \wedge \neg p$ (or something that contradicts a known fact).
- Since this cannot happen, our assumption must be wrong, thus, $\neg q$ is false. q is true.



Proof by Contradiction Example

Prove: If $(3n+2)$ is odd, then n is odd.

Proof:

- Given: $(3n+2)$ is odd.
- Assume that n is not odd, that is n is even.
- If n is even, there is some integer k such that $n=2k$.
- $(3n+2) = (3(2k)+2)=6k+2 = 2(3k+1)$, which is 2 times a number.
- Thus $3n+2$ turned out to be even, but we know it's odd.
- This is a contradiction. Our assumption was wrong.
- Thus, n must be odd.



Proof by Contradiction Example

Prove that the $\sqrt{2}$ is irrational.

- Assume that $\sqrt{2}$ is not irrational, i.e. $\sqrt{2}$ is rational.
- Hence, $\sqrt{2} = \frac{a}{b}$ and a and b have no common factors.
(Rational definition, fraction must be in lowest terms.)
- So $a^2 = 2b^2$ which means a is even, hence $a = 2c$
- Therefore, $b^2 = 2c^2$ then b must be even
- So a and b have at least the common factor 2
- Contradiction, so $\sqrt{2}$ is irrational after all!



Technique #4: Proof by Cases

- Given a problem of the form:
 - $(p_1 \vee p_2 \vee \dots \vee p_n) \rightarrow q$
 - where p_1, p_2, \dots, p_n are the cases
- This is equivalent to the following:
 - $[(p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge \dots \wedge (p_n \rightarrow q)]$
- So prove all the clauses are true.



Proof by Cases Example

- Prove: If n is an integer, then $n^2 \geq n$
 - $(n = 0 \vee n \geq 1 \vee n \leq -1) \rightarrow n^2 \geq n$
- Show for all the three cases, i.e.,
 - $(n = 0 \rightarrow n^2 \geq n) \wedge (n \geq 1 \rightarrow n^2 \geq n)$
 $\wedge (n \leq -1 \rightarrow n^2 \geq n)$



Proof by Cases (cont'd)

- Case 1: Show that $n = 0 \rightarrow n^2 \geq n$
 - When $n=0$, $n^2=0$.
 - $0 \geq 0$ 😊
- Case 2: Show that $n \geq 1 \rightarrow n^2 \geq n$
 - Multiply both sides of the inequality by n
 - We get $n^2 \geq n$



Proof by Cases (cont'd)

- Case 3: Show that $n \leq -1 \rightarrow n^2 \geq n$
- Given $n \leq -1$,
- We know that n^2 cannot be negative, i.e., $n^2 > 0$
- We know that $0 > -1$
- Thus, $n^2 > -1$. We also know that $-1 \geq n$ (given)
- Therefore, $n^2 \geq n$



Proof by Cases Can you finish this?

Theorem: Given two real numbers x and y ,

$$\text{abs}(x*y) = \text{abs}(x)*\text{abs}(y)$$

Exhaustively determine the premises

Case p1: $x \geq 0, y \geq 0$

Proof: $x*y \geq 0$ so $\text{abs}(x*y) = x*y$ and $\text{abs}(x) = x$ and $\text{abs}(y) = y$ so $\text{abs}(x)*\text{abs}(y) = x*y$

Case p2: $x < 0, y \geq 0$

Case p3: $x \geq 0, y < 0$

Case p4: $x < 0, y < 0$