Functions (Rosen, Section 2.3)

TOPICS

• Definition of Function
• Domain, Codomain, Range
• One-to-One Functions
• Increasing Functions

What’s the difference between codomain and range?
Range contains the codomain values that A maps to

Function Definitions

• A function $f$ from sets $A$ to $B$ assigns exactly one element of $B$ to each element of $A$.

• Example: the floor function

<table>
<thead>
<tr>
<th>Domain</th>
<th>Codomain</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>2.4</td>
<td>1</td>
</tr>
<tr>
<td>1.6</td>
<td>2</td>
</tr>
<tr>
<td>5.0</td>
<td>3</td>
</tr>
<tr>
<td>4.8</td>
<td>4</td>
</tr>
<tr>
<td>2.3</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>6</td>
</tr>
</tbody>
</table>

Range: $\{1, 2, 4, 5\}$

Function Definitions

• In Programming
  – Function header definition example

    ```
    int floor(float real) {
    }
    ```

    • Domain = $\mathbb{R}$
    • Codomain = $\mathbb{Z}$
Other Functions

• The identity function, \( f_{id} \), on \( A \) is the function where: \( f_{id}(x) = x \) for all \( x \) in \( A \).
  \( A = \{a, b, c\} \) and \( f(a) = a, f(b) = b, f(c) = c \)

  ■ Successor function, \( f_{succ}(x) = x + 1 \), on \( Z \)
    • \( f(1) = 2 \)
    • \( f(-17) = -16 \)
    • \( f(a) \) Does NOT map to \( b \)

  ■ Predecessor function, \( f_{pred}(x) = x - 1 \), on \( Z \)
    • \( f(1) = 0 \)
    • \( f(-17) = -18 \)

Functions in CS

• What are ceiling and floor useful for?
  – Data stored on disk are represented as a string of bytes. Each byte = 8 bits. How many bytes are required to encode 100 bits of data?

  Need smallest integer that is at least as large as 100/8

  \[ 100/8 = 12.5 \]
  But we don’t work with \( \frac{1}{2} \) a byte.
  So we need 13 bytes

What is NOT a function?

• Consider \( f_{\text{SQRT}}(x) \) from \( Z \) to \( R \).
  • This does not meet the given definition of a function, because \( f_{\text{SQRT}}(16) = \pm 4 \).
  • In other words, \( f_{\text{SQRT}}(x) \) assigns exactly one element of \( Z \) to two elements of \( R \).

  No Way!
  Say it ain’t so!!

Note that the convention is that \( \sqrt{x} \) is always the positive value. \( f_{\text{SQRT}}(x) = \pm \sqrt{x} \)
1 to 1 Functions

- A function $f$ is said to be one-to-one or injective if and only if $f(a) = f(b)$ implies that $a = b$ for all $a$ and $b$ in the domain of $f$.
- Example: the square function from $\mathbb{Z}^+$ to $\mathbb{Z}^+$

1  
2  
3  
4  
...  
9  
...  
16

1 to 1 Functions, cont.

- Is square from $\mathbb{Z}$ to $\mathbb{Z}$ an example?
  - NO!
  - Because $f_{sq}(-2) = 4 = f_{sq}(+2)$!
- Is floor an example?
  \[ \text{INCONCEIVABLE!!} \]
- Is identity an example?
  \[ \text{Unique at last!!} \]

How dare they have the same codomain!

Increasing Functions

- A function $f$ whose domain and co-domain are subsets of the set of real numbers is called increasing if $f(x) \leq f(y)$ and strictly increasing if $f(x) < f(y)$, whenever
  - $x < y$ and
  - $x$ and $y$ are in the domain of $f$.
- Is floor an example?
  \[ 1.5 < 1.7 \text{ and } \text{floor}(1.5) = 1 = \text{floor}(1.7) \]
  \[ 1.2 < 2.2 \text{ and } \text{floor}(1.2) = 1 < 2 = \text{floor}(2.2) \]
- Is square an example?

When mapping $\mathbb{Z}$ to $\mathbb{Z}$ or $\mathbb{R}$ to $\mathbb{R}$:

$\text{square}(-2) = 4 > 1 = \text{square}(1)$ yet $-2 < 1$

How is Increasing Useful?

- Most programs run longer with larger or more complex inputs.
  - Consider the maze:
    - Larger maze may (in the worst case) take longer to get out.
    - Maze with more walls may (in the worst case) take longer to get out.
- Consider looking up a telephone number in the paper directory…

Square is NOT an increasing function UNLESS…

Domain is restricted to positive $\mathbb{R}$.
Cartesian Products and Functions

- A function with multiple arguments maps a Cartesian product of inputs to a codomain.
- Example:
  - `Math.min` maps \((\mathbb{Z} \times \mathbb{Z})\) to \(\mathbb{Z}\)
    
    ```java
    int minVal = Math.min(23, 99);
    ```
  - `Math.abs` maps \((\mathbb{Q} \times \mathbb{Q}^+\) to \(\mathbb{Q}^+\)
    
    ```java
    int absVal = Math.abs(-23);
    ```

Quiz Check

- Are the following functions increasing?
  - \(Z \rightarrow Z\) \(f(x) = x + 5\)
  - \(Z \rightarrow Z\) \(f(x) = 3x - 1\)