Mathematical Induction

Rosen Chapter 5

Motivation

- Suppose we want to prove the predicate P(n): for every positive value of n,

  $1 + 2 + \ldots + n = n(n + 1)/2.$

- We observe P(1), P(2), P(3), P(4). Conjecture: $\forall n \in N, P(n)$.
- Mathematical induction is a proof technique for proving statements of the form

  $\forall n \in N, P(n)$.

Proving P(3)

- Suppose we know: $P(1) \wedge \forall n \geq 1, P(n) \rightarrow P(n + 1)$.
  Prove: P(3)
- Proof:
  1. P(1). [premise]
  2. $P(1) \rightarrow P(2)$. [specialization of premise]
  3. P(2). [step 1, 2, & modus ponens]
  4. $P(2) \rightarrow P(3)$. [specialization of premise]
  5. P(3). [step 3, 4, & modus ponens]

  We can construct a proof for every finite value of n
- Modus ponens: if p and $p \rightarrow q$ then q

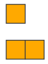
Example: $1 + 2 + \ldots + n = n(n + 1)/2.$

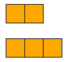
- Verify: F(1): 1(1 + 1)/2 = 1.
- Assume: $F(n) = n(n + 1)/2$
- Show: $F(n + 1) = (n + 1)(n + 2)/2$.

$F(n + 1) = 1 + 2 + \ldots + n + (n + 1)$
$= F(n) + n + 1$
$= n(n + 1)/2 + n + 1$ [Induction hyp.]
$= n(n + 1)/2 + 2(n + 1)/2$
$= (n + 1)(n + 2)/2.$

A Geometrical interpretation

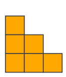
1: 

2: 

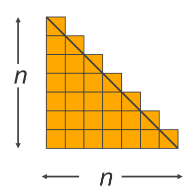
3: 

Put these blocks, which represent numbers, together to form sums:

1 + 2 = 

1 + 2 + 3 = 

A Geometrical interpretation

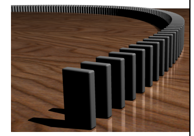


Area is $n^2/2 + n/2 = n(n + 1)/2$

The Principle of Mathematical Induction

- Let P(n) be a statement that, for each natural number n, is either true or false.
- To prove that $\forall n \in N, P(n)$, it suffices to prove:
  - P(1) is true. **(base** case)
  - $\forall n \in N, P(n) \rightarrow P(n+1)$. (inductive **step**)
- This is not magic.
- It is a recipe for constructing a proof for an arbitrary $n \in N$.

Mathematical Induction and the Domino Principle



If

the 1st domino falls over

and

the *n*th domino falls over implies that domino (*n* + 1) falls over

then

domino *n* falls over for all $n \in \mathbf{N}$.

image from http://en.wikipedia.org/wiki/File:Dominoeffect.png

Proof by induction

- 3 steps:
  - Prove P(1). [the basis]
  - Assume P(n) [the induction hypothesis]
  - Using P(n) prove P(n + 1) [the inductive step]

Example

- Show that any postage of ≥ 8¢ can be obtained using 3¢ and 5¢ stamps.
- First check for a few values:

8¢	=	3¢ + 5¢
9¢	=	3¢ + 3¢ + 3¢
10¢	=	5¢ + 5¢
11¢	=	5¢ + 3¢ + 3¢
12¢	=	3¢ + 3¢ + 3¢ + 3¢

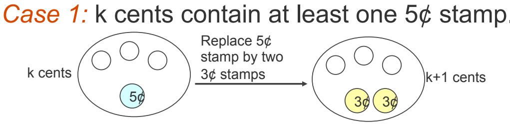
- How to generalize this?

Example

- Let P(n) be the sentence "n cents postage can be obtained using 3¢ and 5¢ stamps".
- Want to show that
  "P(k) is true" *implies* "P(k+1) is true"
  for all k ≥ 8.
- 2 cases:
  1) P(k) is true and
     the k cents contain at least one 5¢.
  2) P(k) is true and
     the k cents do not contain any 5¢.

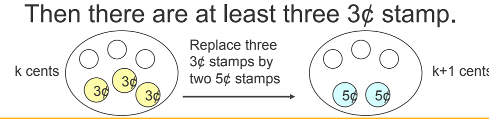
Example

Case 1: k cents contain at least one 5¢ stamp.



Case 2: k cents do not contain any 5¢ stamp.

Then there are at least three 3¢ stamp.



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Examples

- Show that $1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$
- Show that for $n \geq 4$ $2^n < n!$
- Show that $n^3 - n$ is divisible by 3 for every positive n.
- Show that $1 + 3 + 5 + \ldots + (2n+1) = (n+1)^2$
- Prove that a set with n elements has 2^n subsets

All horses have the same color

- *Base case:* If there is only one horse, there is only one color.
- *Induction step:* Assume as induction hypothesis that within any set of n horses, there is only one color. Now look at any set of n + 1 horses. Number them: 1, 2, 3, ..., n, n + 1. Consider the sets {1, 2, 3, ..., n} and {2, 3, 4, ..., n + 1}. Each is a set of only n horses, therefore within each there is only one color. But the two sets overlap, so there must be only one color among all n + 1 horses.
- This is clearly wrong, but can you find the flaw?

Strong induction

- Induction:
  - P(1) is true.
  - $\forall n \in N, P(n) \rightarrow P(n+1)$.
  - Implies $\forall n \in N, P(n)$
- Strong induction:
  - P(1) is true.
  - $\forall n \in N, (P(1) \land P(2) \land \ldots \land P(n)) \rightarrow P(n+1)$.
  - Implies $\forall n \in N, P(n)$

Example

- Prove that all natural numbers ≥ 2 can be represented as a product of primes.
- Basis: 2: 2 is a prime.
- Assume that 1, 2,…, n can be represented as a product of primes.

Example

- Show that n+1can be represented as a product of primes.
  - Case n+1 is a prime: It can be represented as a product of 1 prime, itself.
  - Case n+1 is composite: Then, n + 1 = ab, for some a,b < n + 1.
    - Therefore, $a = p_1p_2 \ldots p_k$ & $b = q_1q_2 \ldots q_l$, where the p_is & q_is are primes.
    - Represent $n+1 = p_1p_2 \ldots p_kq_1q_2 \ldots q_l$.

Induction and Recursion

- Induction is useful for proving correctness/ design of recursive algorithms
- Example

```
// Returns base ^ exponent.
// Precondition: exponent >= 0
public static int pow(int x, int n) {
    if (n == 0) {
        // base case; any number to 0th power is 1
        return 1;
    } else {
        // recursive case:  x^n = x * x^(n-1)
        return x * pow(x, n-1);
    }
}
```

Induction and Recursion

- n! of some integer n can be characterized as:
  n! = 1 for n = 0; otherwise
  n! = n (n - 1) (n - 2) ... 1
- Want to write a recursive method for computing it. We notice that n! = n (n – 1)!
- This is all we need to put together the method:

```
public static int factorial(int n) {
    if (n == 0) {
        return 1;
    } else {
        return n * factorial(n-1);
    }
}
```

More induction examples

- Let n be a positive integer. Show that every 2^n x 2^n checkerboard with one square removed can be tiled using right triominoes, each covering three squares at a time.