Permutations and Combinations

Rosen, Chapter 5.3

Permutations

- In a family of 5, how many ways can we arrange the members of the family in a line for a photograph?

Permutations

- A permutation of a set of distinct objects is an ordered arrangement of these objects.
  - Example: (1, 3, 2, 4) is a permutation of the numbers 1, 2, 3, 4
- How many permutations of n objects are there?

How many permutations?

- How many permutations of n objects are there?
- Using the product rule:

$$n \cdot (n-1) \cdot (n-2) ,\ldots, 2 \cdot 1 = n!$$

The Traveling Salesman Problem (TSP)

TSP: Given a list of cities and their pairwise distances, find a shortest possible tour that visits each city exactly once.

Objective: find a permutation $a_1,\ldots,a_n$ of the cities that minimizes

$$d(a_1, a_2) + d(a_2, a_3) + \ldots + d(a_{n-1}, a_n) + d(a_n, a_1)$$

where $d(i, j)$ is the distance between cities i and j

An optimal TSP tour through Germany's 15 largest cities

Solving TSP

- Go through all permutations of cities, and evaluate the sum-of-distances, keeping the optimal tour.
- Need a method for generating all permutations
- Note: how many solutions to a TSP problem with n cities?

Generating Permutations

- Let's design a recursive algorithm for generating all permutations of {1,2,…,n}.
  - Starting point: decide which element to put first
  - what needs to be done next?
  - what is the base case?

r-permutations

- An ordered arrangement of r elements of a set:
  r-permutations of a set with n elements: **P(n,r)**
- Example: List the 2-permutations of {a,b,c}.
  P(3,2) = 3 x 2 = 6
- Let n and r be integers such that $0 \le r \le n$ then there are
  P(n,r) = n (n – 1)… (n – r + 1)
  r-permutations of a set with n elements. Can be expressed as:
  P(n, r) = n! / (n – r)!
  Note that P(n, 0) = 1.

r-permutations - example

- How many ways are there to select a first-prize winner, a second prize winner and a third prize winner from 100 people who have entered a contest?

Combinations

- How many poker hands (five cards) can be dealt from a deck of 52 cards?
- How is this different than r-permutations?

Combinations

- The number of r-combinations out of a set with n elements is denoted as C(n,r) or $\binom{n}{r}$
  - {1,3,4} is a 3-combination of {1,2,3,4}
  - How many 2-combinations of {a,b,c,d}

r-combinations

- How many r-combinations?

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

Note that C(n, 0) = 1

- C(n,r) satisfies:

$$C(n,r) = C(n,n-r)$$

  - We can see that easily without using the formula

Unordered versus ordered selections

- Two ordered selections are the same if
  - the elements chosen are the same;
  - the elements chosen are in the same order.
- Ordered selections: **r-permutations**.
- Two unordered selections are the same if
  - the elements chosen are the same.
    (regardless of the order in which the elements are chosen)
- Unordered selections: **r-combinations.**

Relationship between P(n,r) and C(n,r)

- Suppose we want to compute P(n,r) .
- Constructing an r-permutation from a set of n elements can be thought as a 2-step process:
  Step 1: Choose a subset of r elements;
  Step 2: Choose an ordering of the r-element subset.
- Step 1 can be done in C(n,r) different ways.
- Step 2 can be done in r! different ways.
- Based on the multiplication rule, P(n,r) = C(n,r) · r!
- Thus

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r! \cdot (n-r)!}$$

r-combinations

- Example: How many bit strings of length n contain exactly r ones?

Example

- The faculty in biology and computer science want to develop a program in computational biology. A committee of 4 composed of two biologists and two computer scientists is tasked with doing this. How many such committees can be assembled out of 20 CS faculty and 30 biology faculty?

Example

- A coin is flipped 10 times, producing either heads or tails. How many possible outcomes
  - are there in total?
  - contain exactly two heads?
  - contain at least three heads?
  - contain the same number of heads and tails?

Some Advice about Counting

- Apply the multiplication rule if
  - The elements to be counted can be obtained through a multistep selection process.
  - Each step is performed in a fixed number of ways regardless of how preceding steps were performed.
- Apply the addition rule if
  - The set of elements to be counted can be broken up into disjoint subsets
- Apply the inclusion/exclusion rule if
  - It is simple to over-count and then to subtract duplicates

Some more advice about Counting

- Make sure that
  1) every element is counted;
  2) no element is counted more than once.
  (avoid double counting)
- When using the addition rule:
  1) every outcome should be in some subset;
  2) the subsets should be disjoint; if they are not, subtract the overlaps

Example using Inclusion/Exclusion Rule

How many integers from 1 through 100 are multiples of 4 or multiples of 7 ?

A: integers from 1 through 100 which are multiples of 4;
B: integers from 1 through 100 which are multiples of 7.
we want to find $|A \cup B|$.

$|A \cup B| = |A| + |B| - |A \cap B|$ (incl./excl. rule)

$A \cap B$ is the set of integers from 1 through 100 which are multiples of 28.

Computing C(n, k) recursively

- consider the nth object

C(n,k) = C(n-1,k-1) + C(n-1,k)
pick n or don't

C(n, k): base case

- C(k, k) = 1
  Why?
- C(n, 0) = 1
  Why?

Recurrence relation

- We can write this as a **recurrence relation**
  - a recursive mathematical expression

C(n,k) = C(n-1,k-1) + C(n-1,k)
pick n or don't
C(k,k) = 1
C(n,0) = 1

and we can code this easily as a recursive method.