Motivating question

In a family of 3, how many ways are there to arrange the members of the family in a line for a photograph?

A) 3 x 3
B) 3!
C) 3 x 3 x 3
D) $2^3$
Permutations

- A permutation of a set of distinct objects is an ordered arrangement of these objects.
  - Example: (1, 3, 2, 4) is a permutation of the numbers 1, 2, 3, 4

- How many permutations of n objects are there?
How many permutations?

- How many permutations of n objects are there?
- Using the product rule:
  \[ n \cdot (n - 1) \cdot (n - 2) \cdot \ldots \cdot 2 \cdot 1 = n! \]
Anagrams

- Anagram: a word, phrase, or name formed by rearranging the letters of another.

Examples:
“cinema” is an anagram of iceman
"Tom Marvolo Riddle" = "I am Lord Voldemort"

The anagram server:  http://wordsmith.org/anagram/
Example

Count the number of ways to arrange n men and n women in a line so that no two men are next to each other and no two women are next to each other.

a) $n!$
b) $n! \cdot n!$
c) $2 \cdot n! \cdot n!$
Example

- You invite 6 people for a dinner party. How many ways are there to seat them around a round table?

  - considering two seatings to be the same if everyone has the same left and right neighbors

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The Traveling Salesman Problem (TSP)

**TSP:** Given a list of cities and their pairwise distances, find a shortest possible tour that visits each city exactly once.

Objective: find a permutation $a_1, \ldots, a_n$ of the cities that minimizes

$$d(a_1, a_2) + d(a_2, a_3) + \ldots + d(a_{n-1}, a_n) + d(a_n, a_1)$$

where $d(i, j)$ is the distance between cities $i$ and $j$
Solving TSP

- Go through all permutations of cities, and evaluate the sum-of-distances, keeping the optimal tour.

- Do we actually need to consider all permutations of n cities?

- Is our algorithm for TSP that considers all permutations of n-1 elements a feasible one for solving TSP problems with hundreds or thousands of cities?
Generating Permutations

- Let's design a recursive algorithm for generating all permutations of \{1,2,\ldots,n\}.

  Starting point: decide which element to put first
  what needs to be done next?
  what is the base case?
r-permutations

- **r-permutation**: An ordered arrangement of r elements of a set.

Example: List the 2-permutations of \{a,b,c\}.

(a,b), (a,c), (b,a), (b,c), (c,a), (c,b)

The number of r-permutations of a set of n elements:

\[ P(n,r) = n(n - 1) \ldots (n - r + 1) \quad (0 \leq r \leq n) \]

Example: \( P(4,2) = 4 \times 3 = 12 \)

Can be expressed as:

\[ P(n, r) = n! / (n - r)! \]

Note that \( P(n, 0) = 1 \).
r-permutations - example

How many ways are there to select a first-prize winner, a second prize winner and a third prize winner from 100 people who have entered a contest?
How many poker hands (five cards) can be dealt from a deck of 52 cards?

How is this different than r-permutations?

In an r-permutation we cared about order. In this case we don’t.
Combinations

- An r-combination of a set is a subset of size r.
- The number of r-combinations out of a set with n elements is denoted as $C(n,r)$ or \( \binom{n}{r} \).

- \{1,3,4\} is a 3-combination of \{1,2,3,4\}.

- How many 2-combinations of \{a,b,c,d\}?
Unordered versus ordered selections

- Two ordered selections are the same if
  - the elements chosen are the same;
  - the elements chosen are in the same order.

- Ordered selections: \textit{r-permutations}.

- Two unordered selections are the same if
  - the elements chosen are the same.
    (regardless of the order in which the elements are chosen)

- Unordered selections: \textit{r-combinations}.
Permutations or combinations?

- Determine if the situation represents a permutation or a combination:
  - In how many ways can three student-council members be elected from five candidates?
  - In how many ways can three student-council members be elected from five candidates to fill the positions of president, vice-president and treasurer?
  - A DJ will play three songs out of 10 requests.

A) Permutations  B) Combinations
Relationship between $P(n,r)$ and $C(n,r)$

- Suppose we want to compute $P(n,r)$.
- Constructing an $r$-permutation from a set of $n$ elements can be thought as a 2-step process:
  
  Step 1: Choose a subset of $r$ elements;
  
  Step 2: Choose an ordering of the $r$-element subset.

- Step 1 can be done in $C(n,r)$ different ways.
- Step 2 can be done in $r!$ different ways.
- Based on the multiplication rule, $P(n,r) = C(n,r) \cdot r!$.
- Thus

$$C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r! \cdot (n - r)!}$$
r-combinations

How many r-combinations?

\[ C(n, r) = \frac{n!}{r!(n - r)!} \]

Note that \( C(n, 0) = 1 \)

Example: How many poker hands (five cards) can be dealt from a deck of 52 cards?

\[ C(52,5) = 52! / (5!47!) \]
r-combinations

- How many r-combinations?

\[ C(n, r) = \frac{n!}{r!(n-r)!} \]

Note that \( C(n, 0) = 1 \)

- \( C(n,r) \) satisfies:

\[ C(n, r) = C(n, n - r) \]

- We can see that easily without using the formula
Combinations or permutations?

- How many bit strings of length $n$ contain exactly $r$ ones? $P(n,r)$ or $C(n,r)$?
Example

- The faculty in biology and computer science want to develop a program in computational biology. A committee of 4 composed of two biologists and two computer scientists is tasked with doing this. How many such committees can be assembled out of 20 CS faculty and 30 biology faculty?
Example

- A coin is flipped 5 times, producing either heads or tails. How many possible outcomes are there in total?
  - contain exactly two heads?
  - contain at least three heads?
  - contain the same number of heads and tails?
Example

- How many permutations of \{a,b,c,d,e,f,g\} end with a?

A) 5!
B) 6!
C) 7!
D) 6 \times 6!
Example

- How many permutations of the letters ABCDEFGH contain the string ABC?
Example

How many 10 character (digits and lowercase/uppercase letters) passwords are possible if

a) characters cannot be repeated?

b) characters can be repeated?
Some Advice about Counting

- Apply the multiplication rule if
  - The elements to be counted can be obtained through a multistep selection process.
  - Each step is performed in a fixed number of ways regardless of how preceding steps were performed.

- Apply the addition rule if
  - The set of elements to be counted can be broken up into disjoint subsets

- Apply the inclusion/exclusion rule if
  - It is simple to over-count and then to subtract duplicates
Some more advice about Counting

- Make sure that
  1) every element is counted;
  2) no element is counted more than once.
    (avoid double counting)

- When using the addition rule:
  1) every outcome should be in some subset;
  2) the subsets should be disjoint; if they are not, subtract the overlaps
Example using Inclusion/Exclusion Rule

How many integers from 1 through 100 are multiples of 4 or multiples of 7?

A: integers from 1 through 100 which are multiples of 4;
B: integers from 1 through 100 which are multiples of 7.

we want to find \(|A \cup B|\).

\[ |A \cup B| = |A| + |B| - |A \cap B| \] (incl./excl. rule)

A \cap B is the set of integers from 1 through 100 which are multiples of 28.
Computing $C(n, k)$ recursively

- consider the $n$th object

\[
C(n,k) = C(n-1,k-1) + C(n-1,k)
\]

pick $n$ or don't
C(n, k): base case

- C(k, k) = 1
  Why?
- C(n, 0) = 1
  Why?
Computing $C(n, k)$ recursively

\[ C(n,k) = C(n-1,k-1) + C(n-1,k) \]

pick \( n \) or don't

$C(k,k) = 1$

$C(n,0) = 1$

we can easily code this as a recursive method!

- This is an example of a recurrence relation, which is a recursive mathematical expression
Combinations

- We just counted the number of choices
- How can we enumerate them all?
  How many combinations (subsets) of k out of 0 – (n-1) can we create?

Notice that this is a generalization of the subset problem from Assign 1
Combinations(5,3)

012   123   234
013   124
014   134
023
024
034

what is the largest digit we can place in the first position? Can you generalize that for $C(n,k)$?
How do we do it?

- place digits $d$ from $lo$ to $hi$ in position $p$

- then recursively place digits $d+1$ to $hi$ in position $p+1$

- hi: $n-k$ for pos $0$, $n-k+1$ for pos $1$, $n-k+p$ for pos $p$

- let's play with the code