Counting: Basics

Rosen, Chapter 5.1-2

Motivation: Counting is useful in CS

- Application domains such as, security, telecom
  - How many password combinations does a hacker need to crack?
  - How many telephone numbers can be supported
- How many steps are needed to solve a problem
  - (time complexity)
- How much space is needed to solve a problem
  - (space complexity)
- Existence proofs
  - Mathematicians may prove that something (useful) exists without giving an algorithm to find it
  - Computer Scientists and engineers may find exact or approximate approaches to find that thing

Four main concepts this week

- Product rule
- Sum rule
- Inclusion-exclusion principle
- Pigeonhole principle

A simple counting problem

- You have 6 pairs of pants and 10 shirts. How many different outfits does this give?

Counting: the product rule

- If there are n_1 ways of doing one task, and for each of way of doing the first task there are n_2 ways of doing a second task, then there are $n_1 n_2$ ways of performing both tasks.

- Example:
  - You have 6 pairs of pants and 10 shirts. How many different outfits does this give?

Relation to Cartesian products

- The Cartesian product of sets A and B is denoted by ***A x B*** and is defined as:
  A x B **=** { (a,b) | a ∈ A and b ∈ B}

- | **A x B** | = | **A** | * | **B** |

Product rule

- Colorado assigns license plates numbers as three uppercase letters followed by three digits. How many license plates numbers are possible?

Iclicker Question #1

- Colorado assigns license plates numbers as three uppercase letters followed by three digits. How many license plates numbers are possible?

A. 26+26+26+10+10+10
B. 26*26*26+10*10*10
C. 26*26*26*10*10*10
D. 42

Iclicker Question #1

- Colorado assigns license plates numbers as three uppercase letters followed by three digits. How many license plates numbers are possible?

A. 26+26+26+10+10+10
B. 26*26*26+10*10*10
C. 26*26*26*10*10*10
D. 42

IClicker Question #2

- A bit is 0 or 1. How many bit strings with 7 digits are there?

A. 2
B. 7
C. 14 (= 2 X 7)
D. 128 (= 2X2X2X2X2X2X2 = 2^7)
E. 49 (= 7X7 = 7^2)

IClicker Question #2 - Answer

- A bit is 0 or 1. How many bit strings with 7 digits are there?

A. 2
B. 7
C. 14 (= 2 X 7)
D. 128 (= 2X2X2X2X2X2X2 = 2^7)
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IClicker question #3

- How many 8 character passwords are there that only use uppercase English letters?

P	A	S	S	W	O	R	D

A. 8
B. 2^{26}
C. 8 X 26
D. 8^{26}
E. 26^8

IClicker Question #3 Answer

- How many 8 character passwords are there that only use uppercase English letters?

P	A	S	S	W	O	R	D

A. 8
B. 2^{26}
C. 8 X 26
D. 8^{26}
E. 26^{8}

IClicker Question #4

- How many 8 character passwords are there that start with 4 lowercase English letters and end with 4 digits?

A. $4^{26} + 4^{10}$
B. 4^{26} X 4^{10}
C. 26^{4} X 10^{4}
D. $26^{4} + 10^{4}$
E. 4X26 + 4X10

IClicker Question #4 Answer

- How many 8 character passwords are there that start with 4 lowercase English letters and end with 4 digits?

A. $4^{26} + 4^{10}$
B. 4^{26} X 4^{10}
C. 26^{4} X 10^{4}
D. $26^{4} + 10^{4}$
E. 4X26 + 4X10

More examples

- How many functions are there from a set with m elements to a set with n elements?

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- How many functions are there from a set with m elements to a set with n elements?

*A function corresponds to a choice of one of the n elements in the codomain (*a set that includes all the possible values of a given function) for each of the m elements in the domain.

*n possibilities * n possibilities* … =* n^m

More examples

- How many one-to-one functions are there from a set with m elements to a set with n elements?
- *One-to-one - A **function** for which every element of the range of the **function** corresponds to exactly **one** element of the domain.*

More examples

- How many one-to-one functions are there from a set with m elements to a set with n elements?

When m > n, there are none.

It is not possible for every element of m to be associated with only one element n when there are more m(s) than n(s).

More examples

- How many one-to-one functions are there from a set with m elements to a set with n elements?

When m > n, there are none.

When m <= n, the following is true:

There are n selections for the first value of m, n-1 for the second, n-2,…,(n-m+1)

*So for a set of 3 to a set of 5, there are 5*4*3*

IClicker Question #5

- How many 8 character passwords are there that use only English lowercase letters, but no letter is repeated?

A. 8 X 26
B. 26 X 25 X 24 X 23 X 22 X 21 X 20 X 19
C. 8^{26}
D. 26^{8}
E. 26 – 8

IClicker Question #5 Answer

- How many 8 character passwords are there that use only English lowercase letters, but no letter is repeated?

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More examples

- Use the product rule to show that the number of different subsets of a finite set S is $2^{|S|}$

More examples

- Use the product rule to show that the number of different subsets of a finite set S is $2^{|S|}$
- Let each element of the set be represented by a bit. The element is either included in the subset (bit set to 1) or not included (bit set to 0). The question now becomes:

How many bit strings with S digits are there?

IClicker Question #6

- It's Saturday and I'm ready to shop for a new car. There are 10 dealerships in the greater Fort Collins metro area. How many possible trips can I take to those 10 dealerships?

A. 10^2
B. 10!
C. 2^{10}
D. 2*10

IClicker Question #6 Answer

- It's Saturday and I'm ready to shop for a new car. There are 10 dealerships in the greater Fort Collins metro area. How many possible trips can I take to those 10 dealerships?

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C. 2^{10}
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A different counting problem

- X has decided to shop at a single store, either in old town or the foothills mall. If X visits old town, X will shop at one of three stores. If X visits the mall, then X will shop at one of two stores. How many ways could X end up shopping?

The Sum Rule

- If a task can be done either in one of n_1 ways or in one of n_2 ways, and none of the n_1 ways is the same as the n_2 ways, then there are $n_1 + n_2$ ways to do the task.
- This is a statement about set theory: if two sets A and B are disjoint then

$$|A \cup B| = |A| + |B|$$

Example

- A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects. No project is on more than one list. How many possible projects are there to choose from?

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23 + 15 + 19 = 57

- *The student can choose from one of 57 different projects.*

IClicker Question #7

- How many 6 or 7 character passwords are there that use only digits?

A. $10^6 + 10^7$
B. $10^6 \times 10^7$
C. $6^{10} + 7^{10}$
D. $6^{10} \times 7^{10}$
E. 10^{13}

IClicker Question #7 Answer

- How many 6 or 7 character passwords are there that use only digits?

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B. $10^6 \times 10^7$
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E. 10^{13}

Recall product rule

```
for(int i=0; i<M; i++) {
  for(int j=0; i<N; j++) {
    System.out.println("Hi");
  }
}
```

- How many times does this print “Hi”?

Recall product rule

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for(int i=0; i<M; i++) {
  for(int j=0; i<N; j++) {
    System.out.println("Hi");
  }
}
```

- How many times does this print "Hi"?
- M*N

IClicker Question #8

- How many passwords are there of length at least one and at most 6 characters, where each character is a digit?

A. 10 X 9 X 8 X 7 X 6 X 5
B. $10 + 10^2 + 10^3 + 10^4 + 10^5 + 10^6$
C. $10 \times 10^2 \times 10^3 \times 10^4 \times 10^5 \times 10^6$
D. 1 X 2 X 3 X 4 X 5 X 6
E. 1 + 2 + 3 + 4 + 5 + 6

IClicker Question #8 Answer

- How many passwords are there of length at least one and at most 6 characters, where each character is a digit?

A. 10 X 9 X 8 X 7 X 6 X 5
B. $10 + 10^2 + 10^3 + 10^4 + 10^5 + 10^6$
C. $10 \times 10^2 \times 10^3 \times 10^4 \times 10^5 \times 10^6$
D. 1 X 2 X 3 X 4 X 5 X 6
E. 1 + 2 + 3 + 4 + 5 + 6

IClicker Question #9

- How many license plates can be made using either two or three uppercase letters followed by two or three digits?

A. 2*3+2*3
B. $(26^2*10^2)+(26^2*10^3)+(26^3*10^2)+(26^3*10^3)$
C. $(26^2+10^2)*(26^2+10^3)*(26^3+10^2)*(26^3+10^3)$
D. $(26^2*10^3)+(26^2*10^3)+(26^3*10^2)+(26^3*10^2)$
E. $(26^2+10^3)*(26^2+10^3)*(26^3+10^2)*(26^3+10^2)$

IClicker Question #9 Answer

- How many license plates can be made using either two or three uppercase letters followed by two or three digits?

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B. $(26^2*10^2)+(26^2*10^3)+(26^3*10^2)+(26^3*10^3)$
C. $(26^2+10^2)*(26^2+10^3)*(26^3+10^2)*(26^3+10^3)$
D. $(26^2*10^3)+(26^2*10^3)+(26^3*10^2)+(26^3*10^2)$
E. $(26^2+10^3)*(26^2+10^3)*(26^3+10^2)*(26^3+10^2)$

Example

- Suppose you need to pick a password that has length 6-8 characters, where each character is an uppercase letter or a digit, and each password must contain at least one digit. How many possible passwords are there?

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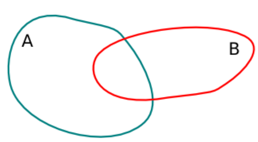
- Suppose you need to pick a password that has length 6-8 characters, where each character is an uppercase letter or a digit, and each password must contain at least one digit. How many possible passwords are there?
- P6 = $36^6 – 26^6$, P7= $36^7 – 26^7$, P7= $36^8 – 26^8$
- Possible passwords = P6+P7+P8

The inclusion exclusion principle

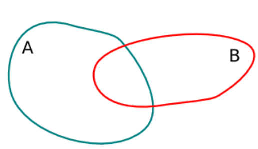
- A more general statement than the sum rule:

$$|A \cup B| = |A| + |B| - |A \cap B|$$



Example

- How many numbers between 1 and 100 are divisible by 2 or 3?



The inclusion exclusion principle

- How many bit strings of length eight start with a 1 **or** end with 00?

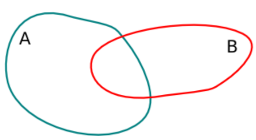
1 - - - - - - - how many?
- - - - - - 0 0 how many?

if I add these
how many have I now counted twice?

The inclusion exclusion principle

- A more general statement than the sum rule:

$$|A \cup B| = |A| + |B| - |A \cap B|$$



The inclusion exclusion principle

- How many bit strings of length eight start with a 1 **or** end with 00?

- $P1 = 2^7$ 1xxxxxxx
- $P2 = 2^6$ xxxxxx00
- $P3 = 2^5$ 1xxxxx00

- $P = P1 + P2 – P3$

IClicker Question #10

- How many passwords of 6 characters, where each character is a lowercase letter, start with two a's **or** end with three b's?

A. $26^2 + 26^3 - 26^5$
B. $2^{26} + 3^{26} - 5^{26}$
C. $4^{26} + 3^{26} - 1^{26}$
D. $26^4 + 26^3 - 26$
E. 26

IClicker Question #10 Answer

- How many passwords of 6 characters, where each character is a lowercase letter, start with two a's **or** end with three b's?

A. $26^2 + 26^3 - 26^5$
B. $2^{26} + 3^{26} - 5^{26}$
C. $4^{26} + 3^{26} - 1^{26}$
D. $26^4 + 26^3 - 26$
E. 26

IClicker Question #11

- How many passwords of 6 characters, where each character is a lowercase letter, start with two a's **and** end with three b's?

A. $26^2 + 26^3 - 26^5$
B. $2^{26} + 3^{26} - 5^{26}$
C. $4^{26} + 3^{26} - 1^{26}$
D. $26^4 + 26^3 - 26$
E. 26

IClicker Question #11 Answer

- How many passwords of 6 characters, where each character is a lowercase letter, start with two a's **and** end with three b's?

A. $26^2 + 26^3 - 26^5$
B. $2^{26} + 3^{26} - 5^{26}$
C. $4^{26} + 3^{26} - 1^{26}$
D. $26^4 + 26^3 - 26$
E. 26

IClicker Question #12

- There is a circular table that seats 4 people. Two seatings are considered to be the same if everyone has the same immediate left and immediate right neighbor. How many different ways can you seat 4 of a group of 10 people around this table?

A. 10^4
B. 4^{10}
C. 10X4
D. 10 X 9 X 8 X 7 = 5040
E. 5040 / 4 = 1260

IClicker Question #12 Answer

- There is a circular table that seats 4 people. Two seatings are considered to be the same if everyone has the same immediate left and immediate right neighbor. How many different ways can you seat 4 of a group of 10 people around this table?

A. 10^4
B. 4^{10}
C. 10X4
D. 10 X 9 X 8 X 7 = 5040
E. 5040 / 4 = 1260

A hairy problem

- In Denver there are two people that have the same number of hairs.

A) True
B) False

On average, non balding people have 90K-150K hairs depending on color. So let's assume the maximum number is less than 300K.

The pigeonhole principle

- If k is a positive integer and k+1 or more objects are placed into k boxes, then there is at least one box containing two or more objects.

Image: http://en.wikipedia.org/wiki/File:TooManyPigeons.jpg

Examples

- How large a group must there be to have at least two with the same birthday ?

Examples

- How large a group must there be to have at least two with the same birthday ?
- It is possible to have 366 people, each with different birthdays. What happens when we add one more person?

Examples

- A drawer contains a dozen brown socks and a dozen black socks, all unmatched. A guy takes socks out at random in the dark.
  - How many socks must he take out to be sure that he has at least two socks of the same color?
  - How many socks must he take out to be sure that he has at least two black socks?

Examples

- A drawer contains a dozen brown socks and a dozen black socks, all unmatched. A guy takes socks out at random in the dark.
  - How many socks must he take out to be sure that he has at least two socks of the same color?
    - 3
  - How many socks must he take out to be sure that he has at least two black socks?
    - He could take out 12 brown socks, then 2 black - 14

IClicker Question #13

- How many cards must you draw before you are guaranteed to have two of the same suit?

A. 2
B. 13
C. 4
D. 5
E. 52

IClicker Question #13 Answer

- How many cards must you draw before you are guaranteed to have two of the same suit?

A. 2
B. 13
C. 4
D. 5
E. 52

IClicker Question #14

- How many cards must you draw before you are guaranteed to have two Aces?

A. 4
B. 5
C. 14
D. 50
E. 52

IClicker Question #14 Answer

- How many cards must you draw before you are guaranteed to have two Aces?

A. 4
B. 5
C. 14
D. 50
E. 52

Examples

- Show that if five different digits between 1 and 8 are selected, there must be at least one pair of these with a sum equal to 9.

- ask yourself: what are the pigeon holes?
  what are the pigeons?

Examples

- Show that if five different digits between 1 and 8 are selected, there must be at least one pair of these with a sum equal to 9.

What are the sets of pairs that add up to 9?
{1,8}, {2,7}, {3,6}, {4,5}

If we choose 5 of these numbers, at least 2 must come from the same set.

Proof question

- 12 students took a CS161 quiz. John Doe made 10 errors. Each of the other students made less than that number. Prove that at least two students made equal number of errors.
- ask yourself: what are the pigeon holes?
  what are the pigeons?

Proof question

- 12 students took a CS161 quiz. John Doe made 10 errors. Each of the other students made less than that number. Prove that at least two students made equal number of errors.

11 students left, 10 possible scores

Proof question

- Assume that in a group of 6 people, each pair of individuals consists of 2 friends or 2 enemies. Show that there are either 3 mutual friends or 3 mutual enemies.
- ask yourself: what are the pigeon holes?
  what are the pigeons?

Proof question

- Assume that in a group of 6 people, each pair of individuals consists of 2 friends or 2 enemies. Show that there are either 3 mutual friends or 3 mutual enemies for any person in the group

- Pigeon holes – enemies or friends
- Pigeons – pairs of people
- Let A be an individual. That individual has 5 people he/she has a relationship with.

IClicker Question #15

- How many cards must you draw to guarantee you have 3 of a kind?

A. 4
B. 9
C. 14
D. 27
E. 52

IClicker Question #15 Answer

- How many cards must you draw to guarantee you have 3 of a kind?

A. 4
B. 9
C. 14
D. 27
E. 52