Mathematical Induction

Rosen Chapter 4.1 (6th edition)
Rosen Ch. 5.1 (7th edition)

Mathmatical Induction

- Mathmatical induction can be used to prove statements that assert that P(n) is true for all positive integers n where P(n) is a propositional function.

- Propositional function – in logic, is a sentence expressed in a way that would assume the value of true or false, except that within the sentence is a variable (x) that is not defined which leaves the statement undetermined.

Motivation

- Suppose we want to prove that for every value of n: $1 + 2 + \ldots + n = n(n + 1)/2$.
- Let $P(n)$ be the predicate
- We observe P(1), P(2), P(3), P(4). Conjecture: $\forall n \in \mathbf{N}, P(n)$.
- Mathematical induction is a proof technique for proving statements of the form

$$\forall n \in \mathbf{N}, P(n).$$

Proving P(3)

- Suppose we know: $P(1) \wedge \forall n \geq 1, P(n) \rightarrow P(n + 1)$.

  Prove: P(3)

- Proof:

  1. P(1). [premise]
  2. $P(1) \rightarrow P(2)$. [specialization of premise]
  3. P(2). [step 1, 2, & modus ponens]
  4. $P(2) \rightarrow P(3)$. [specialization of premise]
  5. P(3). [step 3, 4, & modus ponens]

  We can construct a proof for every finite value of n

- Modus ponens: if p and $p \rightarrow q$ then q

Example: $1 + 2 + \ldots + n = n(n + 1)/2.$

- Verify: F(1) = 1(1 + 1)/2 = 1.
- Assume:

$$1 + 2 + \ldots + n = n(n + 1)/2$$

(Inductive hypothesis)

- Prove: if (P(k) is true, then P(k+1) is true

$1 + 2 + . . . \ + n + (n + 1) = (n+1)(n+2)/2$

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Example: $1 + 2 + \ldots + n = n(n + 1)/2.$

$n(n+1)/2 + (n+1) = (n+1)(n+2)/2$

$n(n+1)/2 + 2(n+1)/2 = (n+1)(n+2)/2$

$(n(n+1)+2(n+1))/2 = (n+1)(n+2)/2$

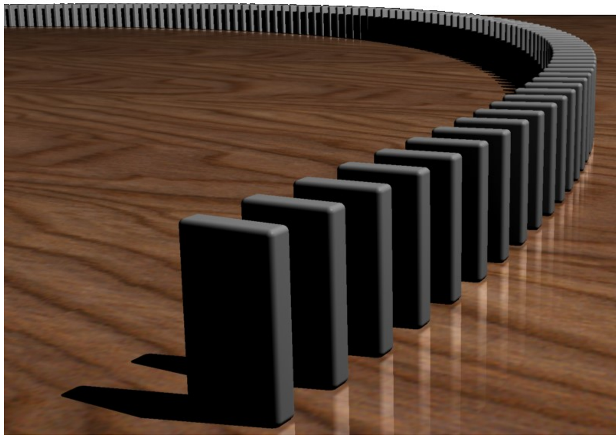
$(n+1)(n+2)/2 = (n+1)(n+2)/2$

Proven!

The Principle of Mathematical Induction

- Let P(n) be a statement that, for each natural number n, is either true or false.
- To prove that $\forall n \in \mathbf{N}, P(n)$, it suffices to prove:
  - P(1) is true. (base case)
  - $\forall n \in \mathbf{N}, P(n) \rightarrow P(n + 1)$. (inductive step)
- This is not magic.
- It is a recipe for constructing a proof for an arbitrary $n \in \mathbf{N}$.

Mathematical Induction and the Domino Principle



If

the 1st domino falls over

and

the nth domino falls over implies that domino ($n + 1$) falls over

then

domino n falls over for all $n \in \mathbf{N}$.

image from http://en.wikipedia.org/wiki/File:Dominoeffect.png

Proof by induction

- 3 steps:
  - Prove P(1). [the basis]
  - Assume P(n) [the induction hypothesis]
  - Using P(n) prove P(n + 1) [the inductive step]

Example

- Show that any postage of ≥ 8¢ can be obtained using 3¢ and 5¢ stamps.
- First check for a few values:

8¢	=	3¢ + 5¢
9¢	=	3¢ + 3¢ + 3¢
10¢	=	5¢ + 5¢
11¢	=	5¢ + 3¢ + 3¢
12¢	=	3¢ + 3¢ + 3¢ + 3¢

- How to generalize this?

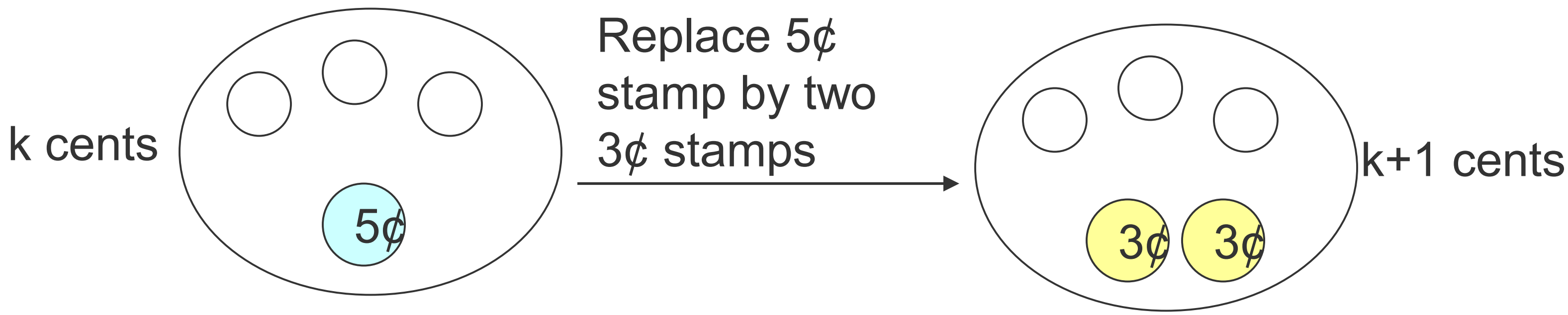
Example

- Let P(n) be the sentence "n cents postage can be obtained using 3¢ and 5¢ stamps".
- Want to show that
  "P(k) is true" *implies* "P(k+1) is true"
  for all k ≥ 8.
- 2 cases:

1) P(k) is true and
   the k cents contain at least one 5¢.
2) P(k) is true and
   the k cents do not contain any 5¢.

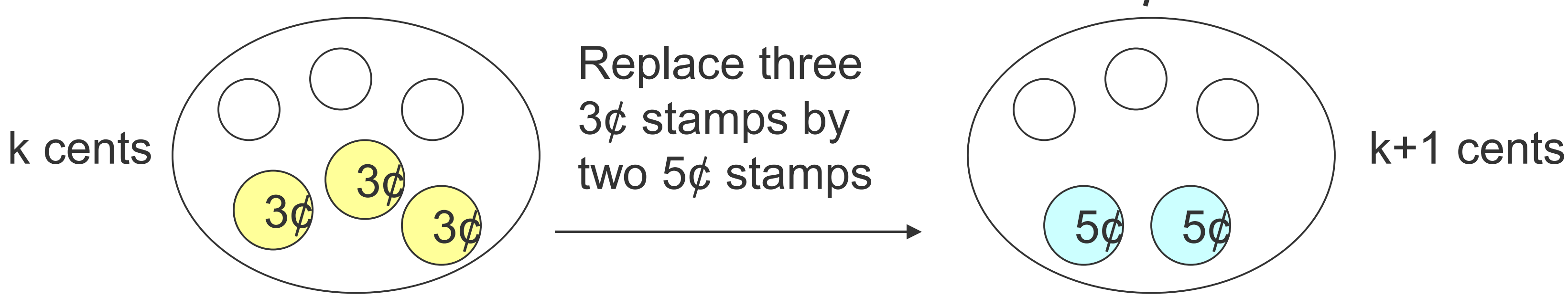
Example

Case 1: k cents contain at least one 5¢ coin.



Case 2: k cents do not contain any 5¢ coin.

Then there are at least three 3¢ coins.



Examples

- Show that $1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$
  - What is the basis statement ? (Note, we want the equation, not a definition)
  - Show that the basis statement is true, completing the base of the induction.
  - What is the inductive hypothesis? (Note, we want the equation, not a definition)
  - What do you need to prove in the inductive step ? (Note, we want the equation, not a description).
  - Complete the inductive step.

Examples

- Show that for $n \geq 4$, $2^n < n!$
  - What is the basis statement ? (Note, we want the equation, not a definition)
  - Show that the basis statement is true, completing the base of the induction.
  - What is the inductive hypothesis? (Note, we want the equation, not a definition)
  - What do you need to prove in the inductive step ? (Note, we want the equation, not a description).
  - Complete the inductive step.

Examples

- Show that $n^3 – n$ is divisible by 3 for $n>0$
  - What is the basis statement ? (Note, we want the equation, not a definition)
  - Show that the basis statement is true, completing the base of the induction.
  - What is the inductive hypothesis? (Note, we want the equation, not a definition)
  - What do you need to prove in the inductive step ? (Note, we want the equation, not a description).
  - Complete the inductive step.

Examples

- Show that $1 + 3 + 5 + ... + (2n+1) = (n+1)^2$
  - What is the basis statement ? (Note, we want the equation, not a definition)
  - Show that the basis statement is true, completing the base of the induction.
  - What is the inductive hypothesis? (Note, we want the equation, not a definition)
  - What do you need to prove in the inductive step ? (Note, we want the equation, not a description).
  - Complete the inductive step.

Examples

- Show that:

  $$a + ar + ar^2 + \ldots + ar^n = \frac{ar^{n+1} - a}{r-1} \text{ when } r <> 1$$

  - What is the basis statement ?   (Note, we want the equation, not a definition)
  - Show that the basis statement is true, completing the base of the induction.
  - What is the inductive hypothesis? (Note, we want the equation, not a definition)
  - What do you need to prove in the inductive step ? (Note, we want the equation, not a description).
  - Complete the inductive step.

Examples

- Show that $7^{n+2} + 8^{2n+1}$ is divisible by 57.
  - What is the basis statement ? (Note, we want the equation, not a definition)
  - Show that the basis statement is true, completing the base of the induction.
  - What is the inductive hypothesis? (Note, we want the equation, not a definition)
  - What do you need to prove in the inductive step ? (Note, we want the equation, not a description).
  - Complete the inductive step.

Wednesday's class

- DO:
- #3 and #5 from the Exercises from section 1 covering mathematical induction from Rosen
- Bring them to class (but do not turn them in).

Examples #3

- Show that $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$ for $n > 0$
  - What is the basis statement ? (Note, we want the equation, not a definition)
  - Show that the basis statement is true, completing the base of the induction.
  - What is the inductive hypothesis? (Note, we want the equation, not a definition)
  - What do you need to prove in the inductive step ? (Note, we want the equation, not a description).
  - Complete the inductive step.

Examples #5

- Show that $1^2 + 3^2 + \ldots + (2n+1)^2 = (n+1)(2n+3)(2n+1)/3$ for $n => 0$
  - What is the basis statement ? (Note, we want the equation, not a definition)
  - Show that the basis statement is true, completing the base of the induction.
  - What is the inductive hypothesis? (Note, we want the equation, not a definition)
  - What do you need to prove in the inductive step ? (Note, we want the equation, not a description).
  - Complete the inductive step.

Be careful!

- Errors in assumptions can lead you to the dark side.

All horses have the same color

- *Base case:* If there is only one horse, there is only one color.
- *Induction step:* Assume as induction hypothesis that within any set of n horses, there is only one color. Now look at any set of n + 1 horses. Number them: 1, 2, 3, ..., n, n + 1. Consider the sets {1, 2, 3, ..., n} and {2, 3, 4, ..., n + 1}. Each is a set of only n horses, therefore within each there is only one color. But the two sets overlap, so there must be only one color among all n + 1 horses.

- **What's wrong here?**

All horses have the same color

- *Base case:* If there is only one horse, there is only one color.
- *Induction step:* Assume as induction hypothesis that within any set of n horses, there is only one color. Now look at any set of n + 1 horses. Number them: 1, 2, 3, ..., n, n + 1. Consider the sets {1, 2, 3, ..., n} and {2, 3, 4, ..., n + 1}. Each is a set of only n horses, therefore within each there is only one color. But the two sets overlap, so there must be only one color among all n + 1 horses.

- **Why must they overlap? That was not one of the assumptions, they are disjoint.**

Theorem

- For every positive integer n, if x and y are positive integers with max (x,y) = n, then x=y.
- Basis: if n = 1, then x = y = 1
- Inductive step: let k be a positive integer. Assume whenever max(x,y) = k and x and y are positive integers, then x=y. Prove max(x,y) = k+1 where x and y are positive integers. max (x-1, y-1) = k, so by inductive hypothesis x-1 = y-1 and x=y.
- What is wrong?

Theorem

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- Basis: if n = 1, then x = y = 1
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- Nothing says x-1 and y-1 are positive integers. X-1 could = 0.

Example

- Show that n+1can be represented as a product of primes.
  - Case n+1 is a prime: It can be represented as a product of 1 prime, itself.
  - Case n+1 is composite: Then, $n + 1 = ab$, for some $a,b < n + 1$.
    - Therefore, $a = p_1p_2 \ldots p_k$ & $b = q_1q_2 \ldots q_l$, where the p_is & q_is are primes.
    - Represent $n+1 = p_1p_2 \ldots p_kq_1q_2 \ldots q_l$.

Induction and Recursion

- Induction is useful for proving correctness/design of recursive algorithms
- Example

```
// Returns base ^ exponent.
// Precondition: exponent >= 0
public static int pow(int x, int n) {
    if (n == 0) {
        // base case; any number to 0th power is 1
        return 1;
    } else {
        // recursive case:  x^n = x * x^(n-1)
        return x * pow(x, n-1);
    }
}
```

Induction and Recursion

- n! of some integer n can be characterized as:

  n! = 1 for n = 0; otherwise

  n! = n (n - 1) (n - 2) ... 1
- Want to write a recursive method for computing it. We notice that n! = n (n – 1)!
- This is all we need to put together the method:

```
public static int factorial(int n) {
    if (n == 0) {
        return 1;
    } else {
        return n * factorial(n-1);
    }
}
```