Counting: Basics

Rosen, Chapter 6.1

Motivation: Counting is useful in CS

- Application domains such as, security, telecom
  - How many password combinations does a hacker need to crack?
  - How many telephone numbers can be supported
- How many steps are needed to solve a problem
  - (time complexity)
- How much space is needed to solve a problem
  - (space complexity)
- Existence proofs
  - Mathematicians may prove that something (useful) exists without giving an algorithm to find it
  - Computer Scientists and engineers may find exact or approximate approaches to find that thing

Four main concepts this week

- Product rule
- Sum rule
- Inclusion-exclusion principle
- Pigeonhole principle

A simple counting problem

- You have 6 pairs of pants and 10 shirts. How many different outfits does this give?

Counting: the product rule

- If there are n_1 ways of doing one task, and for each of way of doing the first task there are n_2 ways of doing a second task, then there are $n_1 n_2$ ways of performing both tasks.
- Example:
  - You have 6 pairs of pants and 10 shirts. How many different outfits does this give?

Relation to Cartesian products

- The Cartesian product of sets A and B is denoted by ***A x B*** and is defined as:

  A x B **=** { (a,b) | a $\in$ A and b $\in$ B}

- | **A x B** | = | **A** | * | **B** |

Product rule

- Colorado assigns license plates numbers as three digits followed by three uppercase letters. How many license plates numbers are possible?

More examples

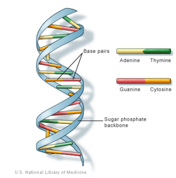
- ~~How many bit strings with 7 digits are there?~~
- How many functions are there from a set with m elements to a set with n elements?
- How many one-to-one functions are there from a set with m elements to a set with n elements?

More examples

- Use the product rule to show that the number of different subsets of a finite set S is $2^{|S|}$
- Product rule and Cartesian products:
  - $|A_1 \times A_2 \times, \ldots, \times A_n| = |A_1| \times |A_2| \times, \ldots, \times |A_n|$

DNA and proteins

- DNA is a long chain that contains one of four nucleotides (A,C,G,T). DNA can code for proteins that are chains of amino acids. There are 20 amino acids. How many nucleotides does it take to code for a single amino acid?



A different counting problem

- X has decided to shop at a single store, either in old town or the foothills mall. If X visits old town, X will shop at one of three stores. If X visits the mall, then X will shop at one of two stores. How many ways could X end up shopping?

The Sum Rule

- If a task can be done either in one of n_1 ways or in one of n_2 ways, and none of the n_1 ways is the same as the n_2 ways, then there are $n_1 + n_2$ ways to do the task.
- This is a statement about set theory: if two sets A and B are disjoint then

$$|A \cup B| = |A| + |B|$$

Example

- A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 4 possible projects. No project is on more than one list. How many possible projects are there to choose from?

Recall product rule

```
for(int i=0; i<M; i++) {
  for(int j=0; i<N; j++) {
    System.out.println("Hi");
  }
}
```

- How many times does this print "Hi"?

Example

- How many license plates can be made using either two or three uppercase letters followed by two or three digits?

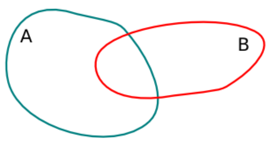
Example

- Suppose you need to pick a password that has length 6-8 characters, where each character is an uppercase letter or a digit, and each password must contain at least one digit. How many possible passwords are there?

The inclusion exclusion principle

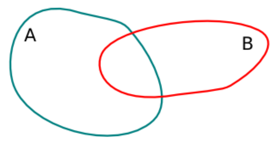
- A more general statement than the sum rule:

$$|A \cup B| = |A| + |B| - |A \cap B|$$



Example

- How many numbers between 1 and 100 are divisible by 2 or 3?



The inclusion exclusion principle

- How many bit strings of length eight start with a 1 **or** end with 00?

1 - - - - - - - how many?
- - - - - - 0 0 how many?

if I add these
how many have I now counted twice?

A hairy problem

- In Denver there are two people that have the same number of hairs.

A) True
B) False

On average, non balding people have 90K-150K hairs depending on color. So let's assume the maximum number is less than 300K.

The pigeonhole principle

- If k is a positive integer and k+1 or more objects are placed into k boxes, then there is at least one box containing two or more objects.

Image: http://en.wikipedia.org/wiki/File:TooManyPigeons.jpg

Examples

- In a group of 367 people, there must be at least two with the same birthday
- A drawer contains a dozen brown socks and a dozen black socks, all unmatched. A guy takes socks out at random in the dark.
  - How many socks must he take out to be sure that he has at least two socks of the same color?
  - How many socks must he take out to be sure that he has at least two black socks?

Examples

- Show that if five different digits between 1 and 8 are selected, there must be at least one pair of these with a sum equal to 9.
- ask yourself: what are the pigeon holes?
  what are the pigeons?

Proof question

- Show that among any n+1 numbers, one can find 2 numbers so that their difference is divisible by n.
- ask yourself: what are the pigeon holes?
  what are the pigeons?

Proof question

- A party is attended by $n \geq 2$ people. Prove that there will always be two people in attendance who have the same number of friends at the party. Assume that the relationship “x-is-a-friend-of-y” is symmetric, i.e., y-is-a-friend-of-x also holds if x-is-a-friend-of-y.