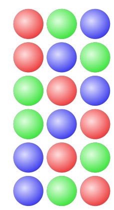
Permutations and Combinations

Rosen, Chapter 5.3

Motivating question

- In a family of 3, how many ways can we arrange the members of the family in a line for a photograph?



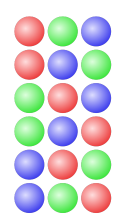
Permutations

- A permutation of a set of distinct objects is an ordered arrangement of these objects.
  - Example: (1, 3, 2, 4) is a permutation of the numbers 1, 2, 3, 4
- How many permutations of n objects are there?

How many permutations?

- How many permutations of n objects are there?
- Using the product rule:

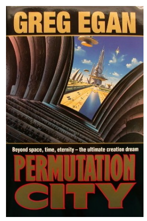
$$n \cdot (n-1) \cdot (n-2) , \ldots, 2 \cdot 1 = n!$$



Anagrams

- Anagram: a word, phrase, or name formed by rearranging the letters of another.

Examples:
"cinema" is an anagram of iceman
"Tom Marvolo Riddle" =
"I am Lord Voldemort"



The anagram server: http://wordsmith.org/anagram/

Example

- Count the number of ways to arrange n men and n women in a line so that no two men are next to each other and no two women are next to each other.

a) n!
b) n! n!
c) 2 n! n!

Example

- You invite 6 people for a dinner party. How many ways are there to seat them around a round table? (Consider two seatings to be the same if everyone has the same left and right neighbors).

A) n!
B) (n-1)!
C) (n+1)!

Example

- In how many ways can a photographer at a wedding arrange six people in a row, (including the bride and groom)?

Example

- In how many ways can a photographer at a wedding arrange six people in a row, including the bride and groom, if the bride is positioned somewhere to the left of the groom?

The Traveling Salesman Problem (TSP)

TSP: Given a list of cities and their pairwise distances, find a shortest possible tour that visits each city exactly once.

Objective: find a permutation $a_1,\ldots,a_n$ of the cities that minimizes

$$d(a_1, a_2) + d(a_2, a_3) + \ldots + d(a_{n-1}, a_n) + d(a_n, a_1)$$

where $d(i, j)$ is the distance between cities i and j

An optimal TSP tour through Germany's 15 largest cities

Solving TSP

- Go through all permutations of cities, and evaluate the sum-of-distances, keeping the optimal tour.
- Need a method for generating all permutations
- Do we actually need to consider all permutations of n cities?

Generating Permutations

- Let's design a recursive algorithm for generating all permutations of {0,1,2,…,n-1}.

Generating Permutations

- Let's design a recursive algorithm for generating all permutations of {0,1,2,…,n-1}.
  - Starting point: decide which element to put first
  - what needs to be done next?
  - what is the base case?

Solving TSP

- Is our algorithm for TSP that considers all permutations of n-1 elements a feasible one for solving TSP problems with hundreds or thousands of cities?

r-permutations

- An ordered arrangement of r elements of a set: number of r-permutations of a set with n elements: **P(n,r)**
- Example: List the 2-permutations of {a,b,c}.

  (a,b), (a,c), (b,a), (b,c), (c,a), (c,b)

  P(3,2) = 3 x 2 = 6
- The number of r-permutations of a set of n elements: then there are

  $P(n,r) = n(n-1)\ldots(n-r+1) \quad (0 \le r \le n)$

  Can be expressed as:

  $P(n, r) = n! / (n-r)!$

  Note that P(n, 0) = 1.

r-permutations - example

- How many ways are there to select a first-prize winner, a second prize winner and a third prize winner from 100 people who have entered a contest?

Question

- How many poker hands (five cards) can be dealt from a deck of 52 cards?

Question

- How many poker hands (five cards) can be dealt from a deck of 52 cards?
- How is this different than r-permutations?

In an r-permutation we cared about order. In this case we don't

Combinations

- An r-combination of a set is a subset of size r
- The number of r-combinations out of a set with n elements is denoted as C(n,r) or $\binom{n}{r}$
  - {1,3,4} is a 3-combination of {1,2,3,4}
  - How many 2-combinations of {a,b,c,d}?

r-combinations

- How many r-combinations?

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

Note that C(n, 0) = 1

- C(n,r) satisfies:

$$C(n,r) = C(n,n-r)$$

  - We can see that easily without using the formula

Unordered versus ordered selections

- Two ordered selections are the same if
  - the elements chosen are the same;
  - the elements chosen are in the same order.
- Ordered selections: **r-permutations**.
- Two unordered selections are the same if
  - the elements chosen are the same.
    (regardless of the order in which the elements are chosen)
- Unordered selections: **r-combinations.**

Relationship between P(n,r) and C(n,r)

- Suppose we want to compute P(n,r) .
- Constructing an r-permutation from a set of n elements can be thought as a 2-step process:
  Step 1: Choose a subset of r elements;
  Step 2: Choose an ordering of the r-element subset.
- Step 1 can be done in C(n,r) different ways.
- Step 2 can be done in r! different ways.
- Based on the multiplication rule, P(n,r) = C(n,r) · r!
- Thus

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r! \cdot (n-r)!}$$

r-combinations

- How many bit strings of length n contain exactly r ones?

Example

- The faculty in biology and computer science want to develop a program in computational biology. A committee of 4 composed of two biologists and two computer scientists is tasked with doing this. How many such committees can be assembled out of 20 CS faculty and 30 biology faculty?

Computing C(n, k) recursively

- consider the nth object

C(n,k) = C(n-1,k-1) + C(n-1,k)
pick n or don't

C(n, k): base case

- C(k, k) = 1
  Why?
- C(n, 0) = 1
  Why?

Computing C(n, k) recursively

C(n,k) = C(n-1,k-1) + C(n-1,k)
pick n or don't
C(k,k) = 1
C(n,0) = 1

we can easily code this as a recursive method!

- This is an example of a **recurrence relation**, which is a recursive mathematical expression

Some Advice about Counting

- Apply the multiplication rule if
  - The elements to be counted can be obtained through a multistep selection process.
  - Each step is performed in a fixed number of ways regardless of how preceding steps were performed.
- Apply the addition rule if
  - The set of elements to be counted can be broken up into disjoint subsets
- Apply the inclusion/exclusion rule if
  - It is simple to over-count and then to subtract duplicates

Some more advice about Counting

- Make sure that
  1) every element is counted;
  2) no element is counted more than once.
     (avoid double counting)
- When using the addition rule:
  1) every outcome should be in some subset;
  2) the subsets should be disjoint; if they are not, subtract the overlaps

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Example using Inclusion/Exclusion Rule

How many integers from 1 through 100 are multiples of 4 or multiples of 7 ?

A: integers from 1 through 100 which are multiples of 4;
B: integers from 1 through 100 which are multiples of 7.
we want to find $|A \cup B|$.

$|A \cup B| = |A| + |B| - |A \cap B|$ (incl./excl. rule)

$A \cap B$ is the set of integers from 1 through 100 which are multiples of 28.