Proof by mathematical induction

You must show your work: the formula you are using, then substitute the numbers into the formula, then calculate the answer. Hand in to your instructor before the beginning of class, and remember that we do not accept late submissions for written assignments. Please answer the questions in the given order to help us in grading.

Prove the following statements using induction. Each proof needs to be complete, including the basis step, and induction step. Each step in the proof needs to be justified (except when it follows by simple algebra). If you are using strong induction, state that explicitly (hint: one of the questions does require it).

1. Prove that $1^3 + 2^3 + \cdots + n^3 = \left\lfloor \frac{n(n+1)}{2} \right\rfloor^2$ for every positive integer $n$.
2. Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ for every positive integer $n$.
3. Prove that for every positive integer $n$,
   
   $1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$

4. Prove that the sum of the first $n$ even positive integers is $n(n + 1)$.
5. Prove that $n^3 + 2n$ is divisible by 3 for every positive integer $n$.
6. Show that you can pay any amount of money greater than $5 using only two-dollar and five-dollar bills.
7. The Fibonacci sequence is defined recursively as follows: $F_1 = 1$, $F_2 = 1$; subsequent elements in the sequence are the sum of the two previous elements: $F_n = F_{n-1} + F_{n-2}$ for $n > 2$. The first few Fibonacci numbers are: 1, 1, 2, 3, 5, 8, 13, 21, \ldots Prove that $F_n \leq 2^n$ for $n \geq 1$.
8. Prove that $F_{4n}$ is divisible by 3, where $F_{4n}$ is the $4n^{th}$ Fibonacci number as defined in the previous question. Hint: express $F_{4n}$ using earlier elements in the series to the point where you can use the induction hypothesis, and the rest is something that is divisible by 3.