Computational Complexity: Measuring the Efficiency of Algorithms

- Rosen Ch. 3.2: Growth of Functions
- Rosen Ch. 3.3: Complexity of Algorithms
- Prichard Ch. 10.1: Efficiency of Algorithms

Algorithm and Computational Complexity

- An ***algorithm*** is a finite sequence of precise instructions for performing a computation for solving a problem.
- ***Computational complexity*** measures the processing time and computer memory required by the algorithm to solve problems of particular size.

Software cost factors

- Human costs
  - Time of developers, testers, maintainers, support team, users
- Managing human costs
  - Adherence to software engineering principles
    - Modularity and Abstraction (separation of concerns principle)
    - Information hiding, good style, readability (design for change principle)

Software cost factors (cont'd)

- Efficiency of algorithms
  - Time to execute algorithms
  - Space required by algorithms
- Focus of this week's lectures

Measuring the efficiency of algorithms

- We have two algorithms: `alg1` and `alg2` that solve the same problem. Our application needs a fast running time.
- How do we choose between the algorithms?

Measuring the efficiency of algorithms

- Implement the two algorithms in Java and compare their running times
- Issues with this approach:
  - How are the algorithms coded? We want to compare the algorithms, not the implementations.
  - What computer should we use? Choice of operations could favor one implementation over another.
  - What data should we use? Choice of data could favor one algorithm over another

Measuring the efficiency of algorithms

- Objective: analyze algorithms independently of specific implementations, hardware, or data
- Observation: An algorithm's execution time is related to the number of operations it executes
- Solution: count the number of significant operations the algorithm will perform for an input of given size

Example: Clicker Q

- Copying an array with n elements requires ____ invocations of copy operations

1. 1
2. n
3. 2n

Example

- Finding the maximum element in a finite sequence

```
public int max (in: array of positive integers a[])
  int max=-1;
   for (int i = 0; i < size_of_array; i++){
        if ( max < a[i] ) max = a[i];
   }
   return max;
}
```

For the input array with size of n integers, `for` loop is executed **n** times.

Example: Clicker Q

- Number of positions to check when running binary search on an array of size 32 when the element is not there:
  1. 1
  2. 2
  3. 5
  4. 32

Growth rates

- Algorithm A requires $n^2 / 2$ operations to solve a problem of size n
- Algorithm B requires $5n+10$ operations to solve a problem of size n
- Which one would you choose?

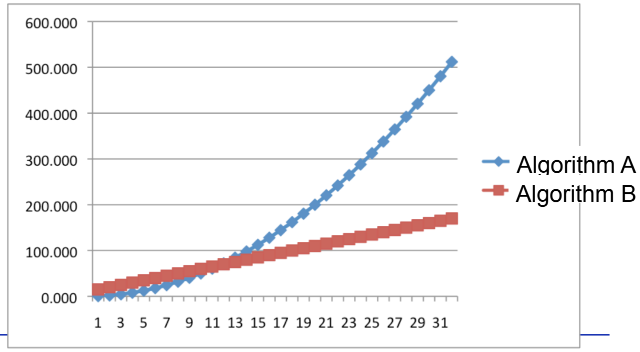
Growth rates

- When we increase the size of input n, how the **execution time grows** for these algorithms?

n	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
$n^2 / 2$	1/2	4/2	9/2	16/2	25/2	36/2	49/2	64/2
5n+10	15	20	45	30	35	40	45	50

n	**50**	**100**	**1,000**	**10,000**	**…**
$n^2 / 2$	1250	5,000	500,000	50,000,000	…
5n+10	260	510	5,010	50,010	…

Growth Rates



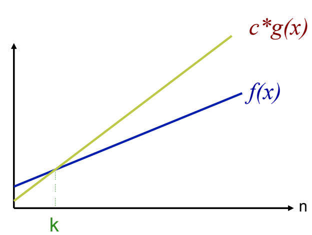
Growth rates

- Algorithm A requires $n^2 / 2$ operations to solve a problem of size n
- Algorithm B requires $5n+10$ operations to solve a problem of size n
- For large enough problem size algorithm B is more efficient
- Important to know how quickly an algorithm's execution time grows as a function of program size
  - We focus on the growth rate:
    - Algorithm A requires time proportional to n^2
    - Algorithm B requires time proportional to n
    - *B's time requirements grows more slowly than A's time requirement (for large n)*

Order of magnitude analysis

- *Big O notation: A function $f(x)$ is $O(g(x))$ if there exist two positive constants, c and k, such that*

$$f(x) \le c*g(x) \qquad \forall\, x > k$$

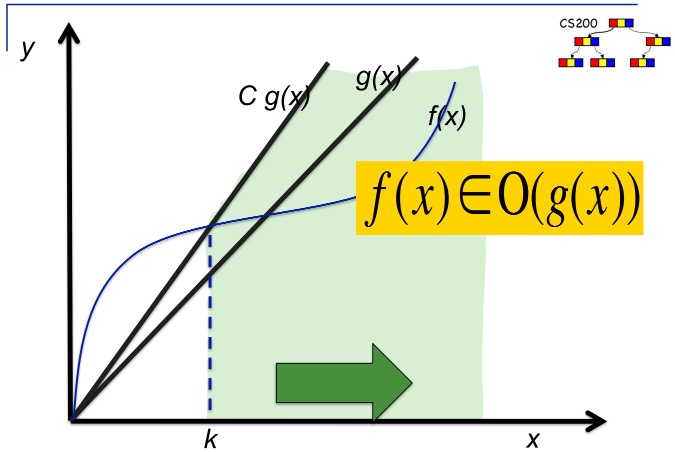


Order of magnitude analysis

- *Big O notation: A function $f(x)$ is $O(g(x))$ if there exist two positive constants, c and k, such that*

$$f(x) \le c*g(x) \qquad \forall\, x > k$$

- Focus is on the shape of the function
  - Ignore the multiplicative constant
- Focus is on large x
  - k allows us to ignore behavior for small x

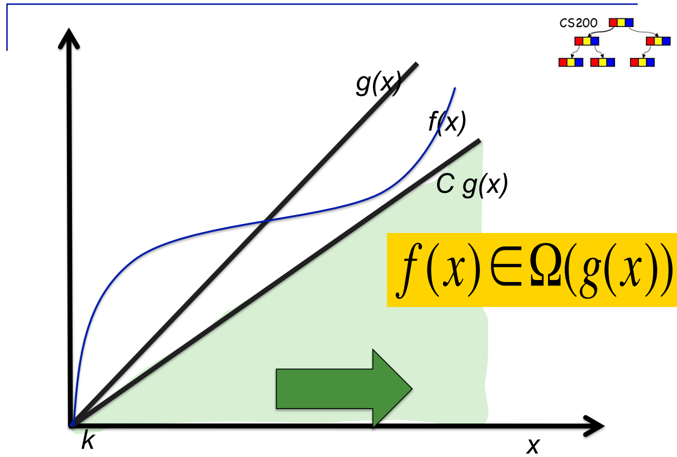


CS200

Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say $f(x)$ is $O(g(x))$ if there are constants C and k such that,

$$|f(x)| \leq C|g(x)|$$

whenever $x > k$

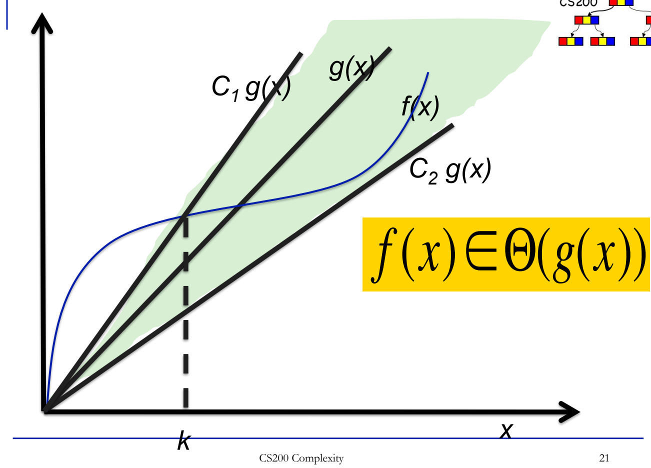


CS200

Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $f(x)$ is $\Omega(g(x))$ if there are positive constants C and k such that,

$$|f(x)| \geq C|g(x)|$$

whenever $x > k$



Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that

$f(x)$* is Θ *$(g(x))$

if *$f(x)$* is O *$(g(x))$* and

$f(x)$* is Ω *$(g(x))$

Order of magnitude analysis

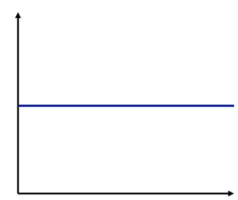
- *Big O notation: A function $f(x)$ is $O(g(x))$ if there exist two positive constants, c and k, such that*

  $$f(x) \le c * g(x) \qquad \forall\, x > k$$

- c and k are witnesses to the relationship that *$f(x)$* is *$O(g(x))$*
- If there is one pair of witnesses (c,k) then there are infinitely many.

Common Shapes: Constant

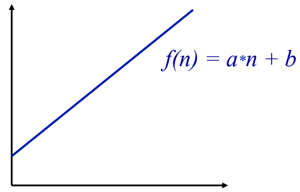
- *$O(1)$*



- examples?

Common Shapes: Linear

- *O(n)*

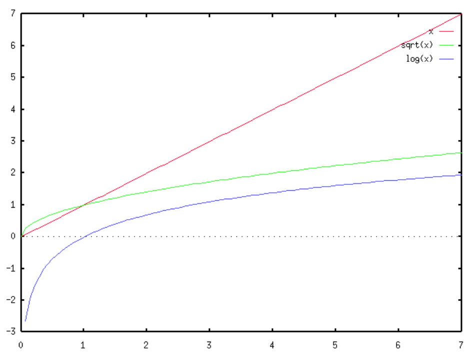


Linear

Example: copying an array

```
for (int i = 0; i < a.size; i++){
    a[i] = b[i];
}
```


Other Shapes: Sublinear



Common Shapes: logarithm

- $log_b n$ is the number x such that $b^x = n$

  $2^3 = 8$ $\log_2 8 = 3$

  $2^4 = 16$ $\log_2 16 = 4$

- $log_b n$: (# of digits to represent n in base b) – 1
- We usually work with base 2

Logarithms (cont.)

- Properties of logarithms
  - $log(x\ y) = log\ x + log\ y$
  - $log(x^a) = a\ log\ x$
  - $log_a n = log_b n\ /\ log_b a$
- logarithm is a **very** slow-growing function
- examples of logarithmic complexity?

Quadratic

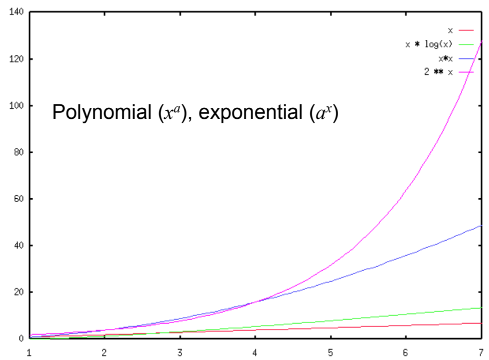
$O(n^2)$:

n times

```
for (int i=0; i < n; i++){
   for (int j=0; j < n; j++) {
      ...
   }
}
```

n times

Other Shapes: Superlinear



Big-O for Polynomials

Theorem: Let

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

where $a_n, a_{n-1} \ldots, a_1, a_0$ are real numbers.
Then $f(x)$ is $O(x^n)$

Example: $x^2 + 5x$ is $O(x^2)$

Clicker Q

Give as good a Big O estimate as possible for the following growth function.

$f(n) = (3n^2 + 8)(n + 1)$

(a) O(n)
(b) $O(n^3)$
(c) $O(n^2)$
(d) O(1)

Combinations of Functions

- Additive Theorem:

  Suppose that $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$.
  Then $(f_1 + f_2)(x)$ is $O(\max(|g_1(x)|,|g_2(x)|))$.

- Multiplicative Theorem:

  Suppose that $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$.
  Then $(f_1 f_2)(x)$ is $O(g_1(x)g_2(x))$.

Practical Analysis - Combinations

- Sequential
  - Big-O bound: Steepest growth dominates
  - Example: copying of array, followed by binary search
    - n + log(n) O(?)
- Embedded code
  - Big-O bound multiplicative
  - Example: a for loop with n iterations and a body taking O(log n) O(?)

Worst and Average Case Time Complexity

- Worst case
  - just how bad can it get: the maximal number of steps
  - our focus in this course
- Average case
  - amount of time expected "usually"
  - In this course we will hand wave when it comes to average case
- Best case
  - The smallest number of steps
- Example: searching for an item in an unsorted array

Practical Analysis - Loops

```
1   public void insertElementAt(Object obj, int index) {
      …
2     for (i = elementCount; i > index; i--) {
3         elementData[i] = elementData[i-1];
      }
      …
    }
```

How many times will line 3 repeat?
On what does the number depend?

Practical Analysis – Dependent loops

```
....
for (i = 0; i < n; i++) {
      for (j = 0; j < i; j++) {
          ...
      }
}
...
```

i = 0: inner-loop iters =0

i = 1: inner-loop iters =1

:

i = n-1: inner-loop iters =n-1

Total = 0 + 1 + 2 + ... + (n-1)

$f(n) = n*(n-1)/2$

$O(n^2)$

Practical Analysis - Recursion

- Number of operations depends on :
  - number of calls
  - work done in each call
- Examples:
  - factorial: how many recursive calls?
  - binary search?
- We will devote more time to analyzing recursive algorithms later in the course.

Final Comments

- Order-of-magnitude analysis focuses on large problems
- If the problem size is always small, you can probably ignore an algorithm's efficiency
- Weigh the trade-offs between an algorithm's time requirements and its memory requirements, expense of programming/maintenance…