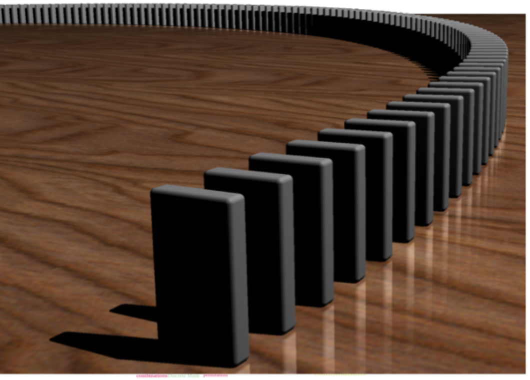
CS 220: Discrete Structures and their Applications

Mathematical Induction

6.4 – 6.6 in zybooks



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Why induction?

Prove algorithm correctness (CS320 is full of it)

The inductive proof will sometimes point out an algorithmic solution to a problem

Strongly connected to recursion

Motivation

Show that any postage of ≥ 8¢ can be obtained using 3¢ and 5¢ stamps.

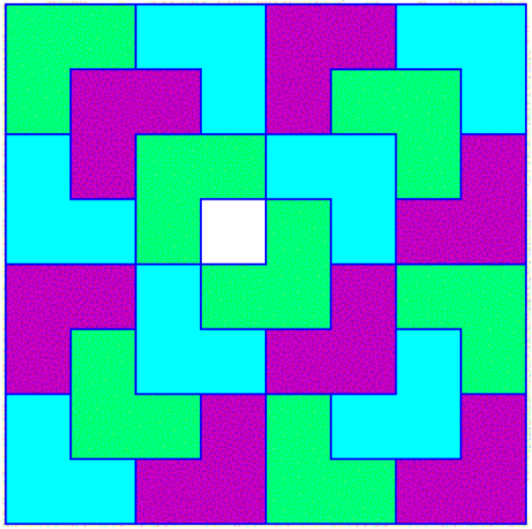
First check for a few values:

8¢	=	3¢ + 5¢
9¢	=	3¢ + 3¢ + 3¢
10¢	=	5¢ + 5¢
11¢	=	5¢ + 3¢ + 3¢
12¢	=	3¢ + 3¢ + 3¢ + 3¢

How to generalize this?

Motivation

Let n be a positive integer. Show that every $2^n \times 2^n$ chessboard with one square removed can be tiled using triominoes, each covering three squares at a time.



Motivation

Prove that for every positive value of n,

$$1 + 2 + ,\ldots, + n = n(n + 1)/2.$$

Motivation

Many mathematical statements have the form:

$\forall n \in \mathbf{N}, P(n)$ $P(n)$: Logical predicate

Example: For every positive value of n,

$1 + 2 + ,\ldots, + n = n(n + 1)/2.$

Predicate – propositional function that depends on a variable, and has a truth value once the variable is assigned a value.

Mathematical induction is a proof technique for proving such statements

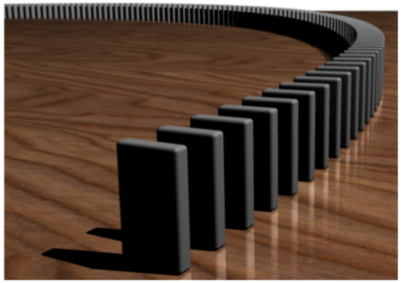
Proving P(3)

Suppose we know:

1. P(1) and 2. $P(n) \rightarrow P(n+1)\ \forall n \geq 1$.

Prove: P(3)

Proof:



1. P(1).	[premise]
2. $P(1) \rightarrow P(2)$.	[specialization of premise]
3. P(2).	[step 1, 2, & modus ponens]
4. $P(2) \rightarrow P(3)$.	[specialization of premise]
5. P(3).	[step 3, 4, & modus ponens]

We can construct a proof for every finite value of n

Modus ponens: if p and $p \rightarrow q$ then q

Example

Theorem: For every positive integer n,

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$

Proof.

By induction on n.

Base case: n = 1.

When n = 1, the left side of the equation is $\sum_{j=1}^{1} j = 1$.

When n = 1, the right side of the equation is 1(1 + 1)/2 = 1.

Therefore, $\sum_{j=1}^{1} j = \frac{1(1+1)}{2}$.

Inductive step: Suppose that for positive integer k, $\sum_{j=1}^{k} j = \frac{k(k+1)}{2}$, then we will show that

$$\sum_{j=1}^{k+1} j = \frac{(k+1)(k+2)}{2}$$

Starting with the left side of the equation to be proven:

$$\sum_{j=1}^{k+1} j = \sum_{j=1}^{k} j + (k+1) \qquad \text{by separating out the last term}$$

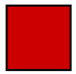
$$= \frac{k(k+1)}{2} + (k+1) \qquad \text{by the inductive hypothesis}$$

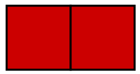
$$= \frac{k(k+1) + 2(k+1)}{2}$$

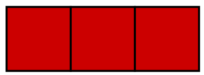
$$= \frac{(k+2)(k+1)}{2} \qquad \text{by algebra}$$

Therefore, $\sum_{j=1}^{k+1} j = \frac{(k+1)(k+2)}{2}$. ■

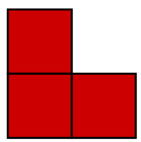
A Geometrical interpretation

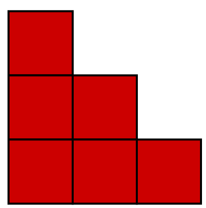
1: 

2: 

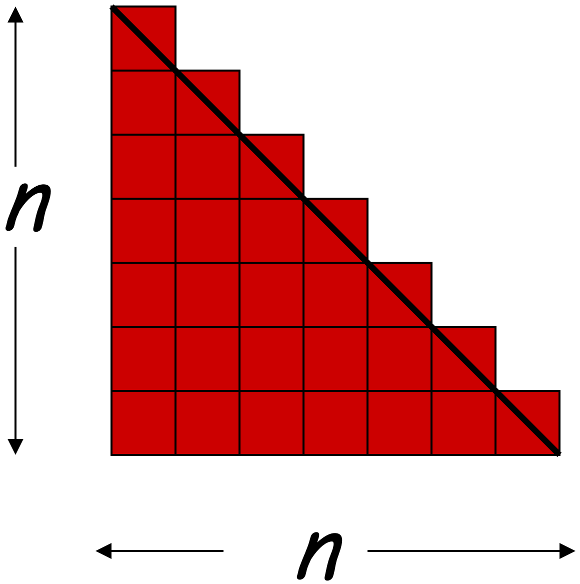
3: 

Put these blocks, which represent numbers, together to form sums:

1 + 2 = 

1 + 2 + 3 = 

A Geometrical interpretation



Area is $n^2/2 + n/2 = n(n + 1)/2$

The principle of mathematical induction

Let P(n) be a statement that, for each natural number n, is either true or false.

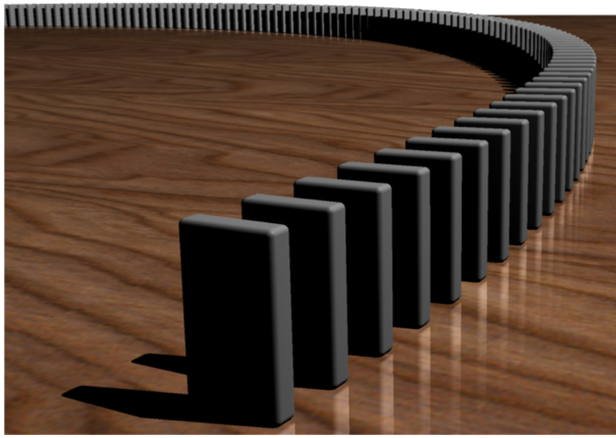
To prove that $\forall n \in \mathbf{N}, P(n)$, it suffices to prove:

- P(1) is true. (**basis** step)
- $\forall n \in \mathbf{N}, P(n) \rightarrow P(n+1)$. (**inductive** step)

This is not magic.

It is a recipe for constructing a proof for an arbitrary $n \in \mathbf{N}$.

the domino principle



If

the 1^{st} domino falls over

and

the nth domino falls over implies that domino $(n + 1)$ falls over

then

domino n falls over for all $n \in \mathbf{N}$.

image from http://en.wikipedia.org/wiki/File:Dominoeffect.png

proof by induction

3 steps:

- Prove P(1). [the basis step]
- Assume P(k) [the induction hypothesis]
- Using P(k) prove P(k + 1) [the inductive step]

Example

- Show that any postage of ≥ 8¢ can be obtained using 3¢ and 5¢ stamps.
- Basis step:

  8¢ = 3¢ + 5¢

Example

Let P(n) be the statement "n cents postage can be obtained using 3¢ and 5¢ stamps".

Want to show that

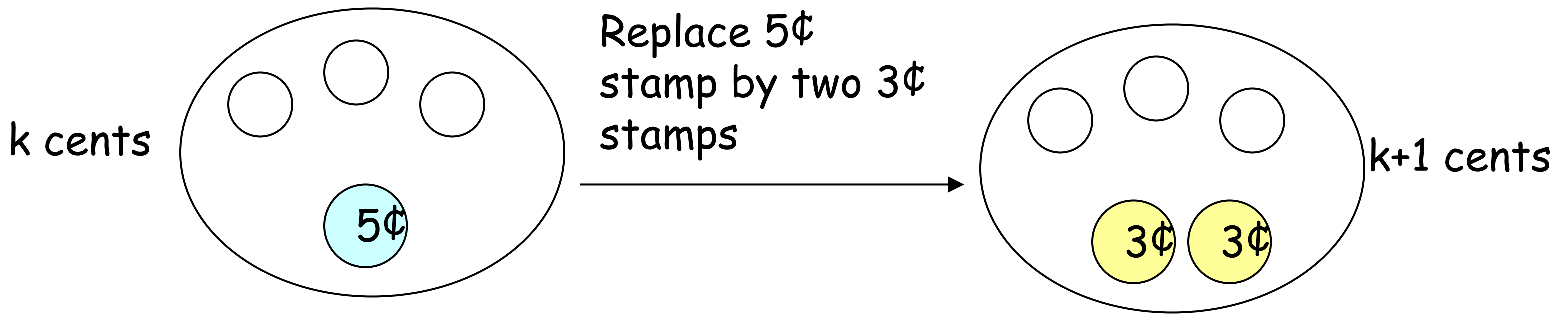
"P(k) is true" *implies* "P(k+1) is true" for all $k \geq 8$.

2 cases:

1) P(k) is true and
   the k cents contain at least one 5¢.
2) P(k) is true and
   the k cents do not contain any 5¢.

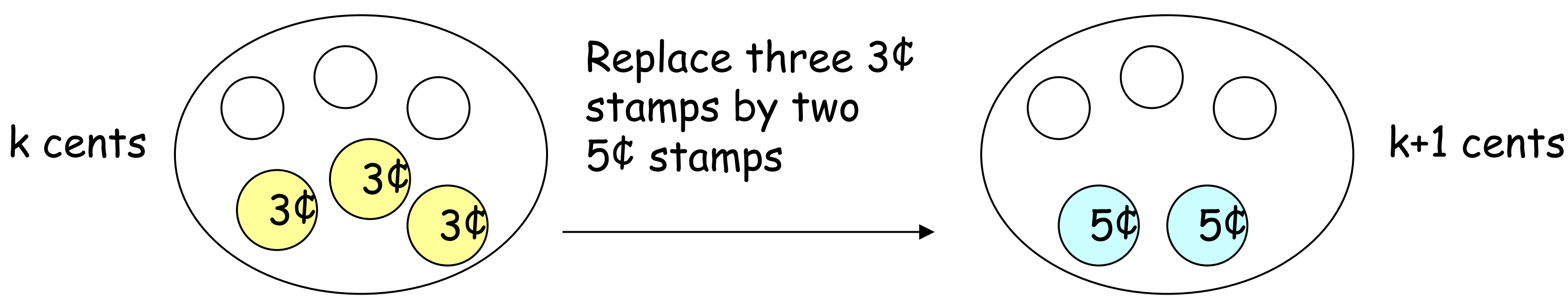
Example

Case 1: k cents contain at least one 5¢ stamp.



Case 2: k cents do not contain any 5¢ stamp.

Then there are at least three 3¢ stamp.



Arithmetic sequences

Sum of an arithmetic sequence: For any integer n ≥ 1:

$$\sum_{j=0}^{n-1}(a + jd) = an + \frac{d(n-1)n}{2}$$

Proof: By induction on n

Base case: n=1

Induction step:

Assume:

$$\sum_{j=0}^{k-1}(a + jd) = ak + \frac{d(k-1)k}{2}$$

Need to prove:

$$\sum_{j=0}^{k}(a + jd) = a(k+1) + \frac{dk(k+1)}{2}$$

Examples

Prove that $1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$

Prove that for $n \geq 4$ $2^n < n!$

Prove that $n^3 - n$ is divisible by 3 for every positive n.

Prove that $1 + 3 + 5 + \ldots + (2n+1) = (n+1)^2$

Prove that a set with n elements has 2^n subsets

Prove that $1^2 + 2^2 + 3^2 + \ldots + n^2 = n(n+1)(2n+1)/6$ for $n>0$

Hint: $2n^2+7n+6 = (n+2)(2n+3)$

each time ask yourself

1. BASE
   - What is the base case? Can I prove the base case?
2. STEP
   - What is the hypothesis?
   - Obligation: What do I need to prove the inductive step
   - How do I complete the inductive step?

Example

Prove that $P(n): 1^2 + 2^2 + 3^2 + \ldots + n^2 = n(n+1)(2n+1)/6$ for $n>0$

Base: $n=1$ $1^2 = 1.2.3/6$

Hypothesis: $P(k): 1^2 + 2^2 + 3^2 + \ldots + k^2 = k(k+1)(2k+1)/6$

Obligation $P(k) \rightarrow P(k+1) : 1^2 + 2^2 + 3^2 + \ldots + (k+1)^2 = (k+1)(k+2)(2k+3)/6$

Proof:

$1^2 + 2^2 + 3^2 + \ldots + (k+1)^2 =$

$1^2 + 2^2 + 3^2 + \ldots + k^2 \quad + (k+1)^2 =$ hypothesis

$k(k+1)(2k+1)/6 + (k+1)^2 = (k+1)(k(2k+1)+6(k+1))/6 =$

$(k+1)(2k^2+7k+6)/6 =$ Hint: $2n^2+7n+6 = (n+2)(2n+3)$

$(k+1)(k+2)(2k+3)/6$

All horses have the same color

Base case: If there is only one horse, there is only one color.

Induction step: Assume as induction hypothesis that within any set of n horses, there is only one color. Now look at any set of n + 1 horses. Number them: 1, 2, 3, ..., n, n + 1. Consider the sets {1, 2, 3, ..., n} and {2, 3, 4, ..., n + 1}. Each is a set of only n horses, therefore within each there is only one color. But the two sets overlap, so there must be only one color among all n+1 horses.

This is clearly wrong, but can you find the flaw?

NOT all horses have the same color

The step from

k = 1 {1}

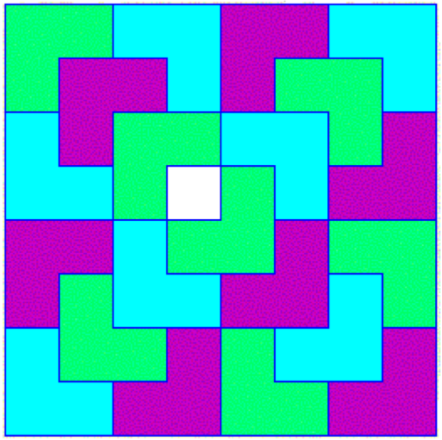
to

k = 2 {1,2}

Fails: there is no intersection: {1} ∩ {2} in a set of two horses, as was incorrectly used in the "proof".

More induction examples

Let n be a positive integer. Show that every $2^n \times 2^n$ chessboard with one square removed can be tiled using L-shaped triominoes, each covering three squares at a time.



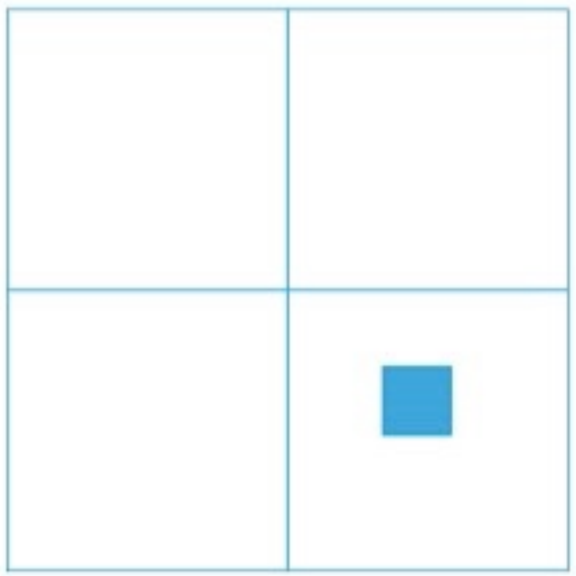
First, show that

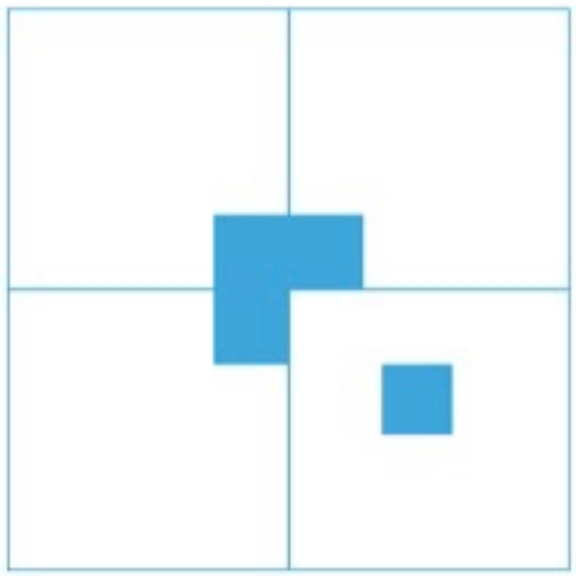
$3 \mid 2^n \times 2^n - 1$

(i.e. 3 divides $2^n \times 2^n - 1$)

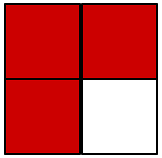
Tiling with triominoes

Divide the board into four sub-boards:





Base case?



A bound on Fibonacci numbers

The Fibonacci sequence:

$$f_0 = 0, f_1 = 1$$

$$f_n = f_{n-1} + f_{n-2} \; for \;\; n \geq 2$$

Theorem: $f_n \leq 2^n$ **for** $n \geq 0$

Strong induction

Induction:

- P(1) is true.
- $\forall n \in N, P(n) \rightarrow P(n + 1)$.
- Implies $\forall n \in N, P(n)$

Strong induction:

- P(1) is true.
- $\forall n \in N, (P(1) \wedge P(2) \wedge \ldots \wedge P(n)) \rightarrow P(n + 1)$.
- Implies $\forall n \in N, P(n)$

Example

Prove that all natural numbers ≥ 2 can be represented as a product of primes.

Basis: $n=2$: 2 is a prime.

Example

Inductive step: show that n+1 can be represented as a product of primes.

- If n+1 is a prime: It can be represented as a product of 1 prime, itself.
- If n+1 is composite: Then, $n + 1 = ab$, for some $a,b < n + 1$.
  - Therefore, $a = p_1p_2 \ldots p_k$ & $b = q_1q_2 \ldots q_l$, where the p_is & q_is are primes.
  - Represent $n+1 = p_1p_2 \ldots p_kq_1q_2 \ldots q_l$.

Breaking chocolate

Theorem: Breaking up a chocolate bar with n "squares" into individual squares takes n-1 breaks.
(Break = dividing (sub) bar in 2 along a "break line")

A **full binary tree** (sometimes proper **binary tree** or 2-**tree**) is a **tree** in which every node other than the leaves has two children.

Prove:

A full binary tree with n leaves has n-1 internal nodes

What is the relation with the chocolate bar?

Induction and the well ordering principle

The well-ordering principle:

any non-empty subset of the non-negative integers has a smallest element.

Surprising fact:

Well-ordering implies the principle of mathematical induction

Smallest element: base

Next element: step