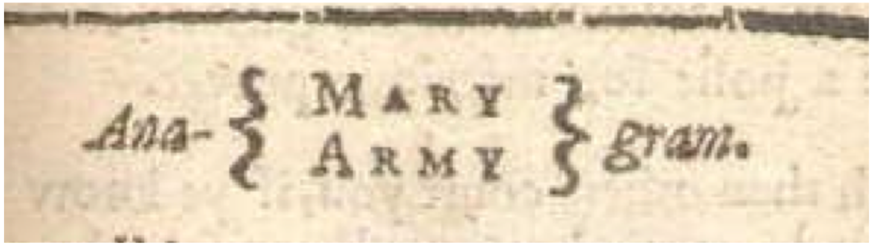
CS 220: Discrete Structures and their Applications

Permutations and combinations

zybooks 7.4-7.6



Motivating question

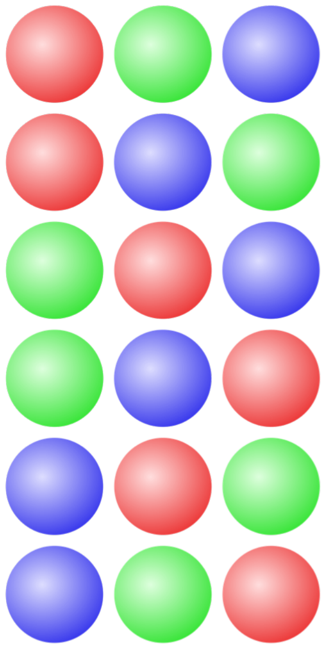
In a family of 3, how many ways are there to arrange the members of the family in a line for a photograph?

A) 3×3

B) $3!$

C) $3 \times 3 \times 3$

D) 2^3



The Traveling Salesman Problem (TSP)

TSP: Given a list of cities and their pairwise distances, find a shortest possible tour that visits each city exactly once.

Objective: find an ordering $a_1,\ldots,a_n$ of the cities that minimizes

$$d(a_1, a_2) + d(a_2, a_3) + \ldots + d(a_{n-1}, a_n) + d(a_n, a_1)$$

where $d(i, j)$ is the distance between cities i and j

An optimal TSP tour through Germany's 15 largest cities

Permutations

A permutation of a set of distinct objects is an ordered arrangement of these objects.

- Example: (1, 3, 4, 2) is a permutation of the numbers 1, 2, 3, 4

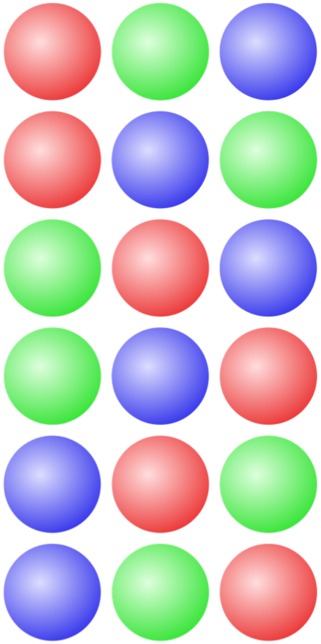
How many permutations of n objects are there?

How many permutations?

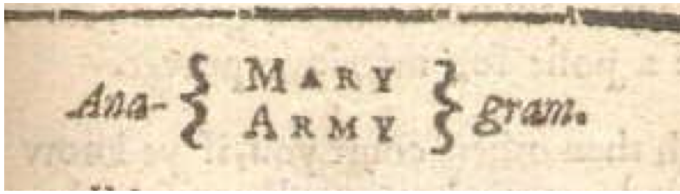
How many permutations of n objects are there?

Using the product rule:

$$n \cdot (n-1) \cdot (n-2) ,\ldots, 2 \cdot 1 = n!$$



Anagrams



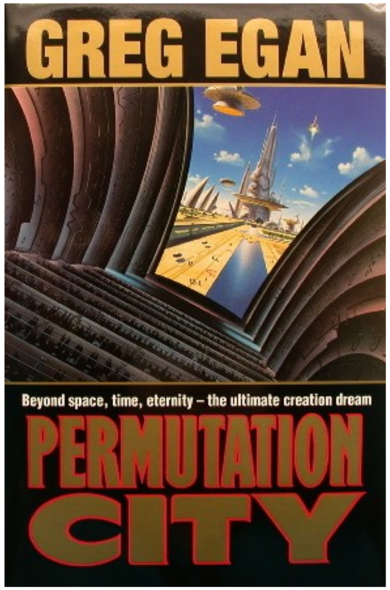
Anagram: a word, phrase, or name formed by rearranging the letters of another.

Examples:

"cinema" is an anagram of iceman

"Tom Marvolo Riddle" is an anagram of "I am Lord Voldemort"

The anagram server: http://wordsmith.org/anagram/



Example

How many permutations of {a,b,c,d,e,f,g} end with a?

A) 5!
B) 6!
C) 7!
D) $6 \times 6!$

Example

You invite 6 people for a dinner party. How many ways are there to seat them around a round table? (Consider two seatings to be the same if everyone has the same left and right neighbors).

A) 6!

B) 5!

C) 7!

Example

Count the number of ways to arrange n men and n women in a line so that no two men are next to each other and no two women are next to each other.

a) $n!$

b) $n!\, n!$

c) $2\, n!\, n!$

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solving TSP

An algorithm for the TSP problem:

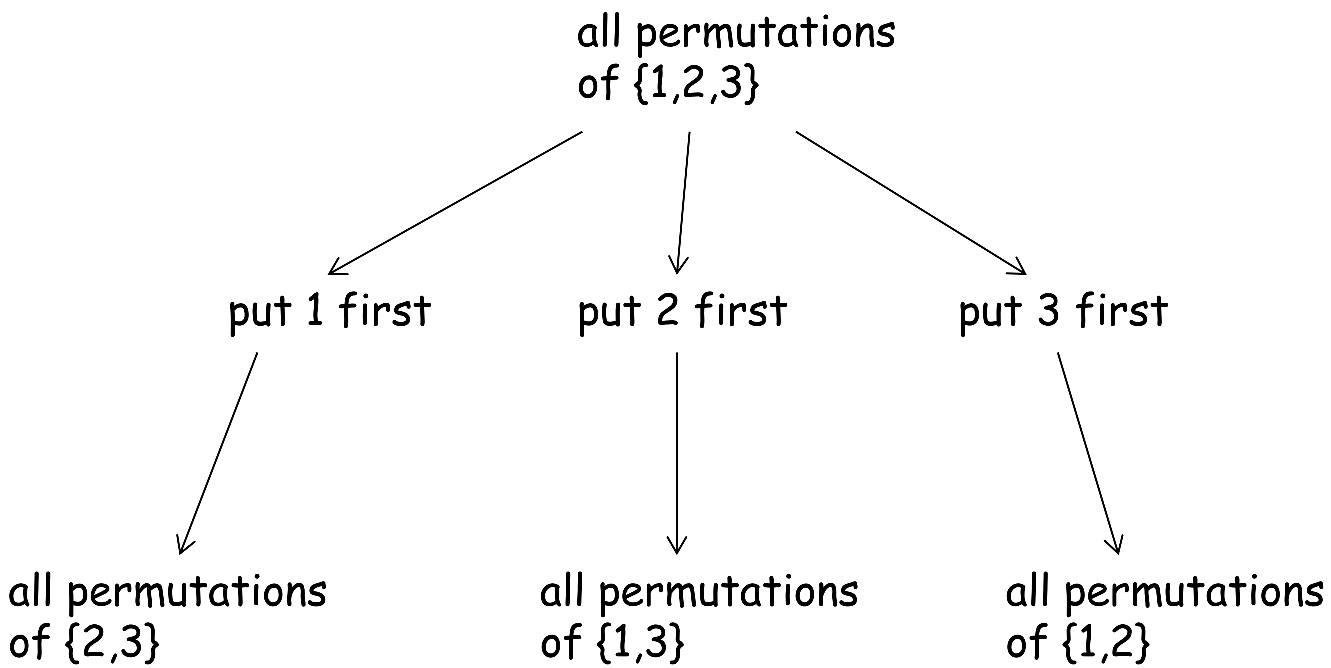
Go through all permutations of cities, and evaluate the sum-of-distances, keeping the optimal tour.

generating permutations

how to generate permutations recursively?

generating permutations

how to generate permutations recursively



generating permutations code

Does this work?

```
def perm(A,f):
  if f == len(A)-1:
    print(A)
  else:
    for i in range(f,len(A)) :
      A[i], A[f] = A[f], A[i]
      perm(A,f+1)

A = []
for i in range(n):
   A.append(i)
perm(A,0)
```

Let's try it ...

solving TSP

Is our algorithm for TSP that considers all permutations a feasible one for solving TSP problems with hundreds or thousands of cities?

NO: 50 cities:

(n-1)!/2 = 12,413,915,592,536,072,670,862,289,047,373,375,038,521,486,354,677,760,000,000,000

We call problems like TSP **intractable.**

Question: would there be a faster algorithm for printing all permutations?

r-permutations

r-permutation: An ordered arrangement of r elements of a set.

Example: List the 2-permutations of {a,b,c}.

(a,b), (a,c), (b,a), (b,c), (c,a), (c,b)

The number of r-permutations of a set of n elements:

$$P(n,r) = n(n-1)\ldots(n-r+1) \qquad (0 \le r \le n)$$

Example: $P(3,2) = 3 \times 2 = 6$

Can be expressed as:

$$P(n, r) = n! / (n-r)!$$

Note that $P(n, 0) = 1$.

r-permutations - example

How many ways are there to select a first-prize winner, a second prize winner and a third prize winner from 100 people who have entered a contest?

permutations with repetitions

How many ways are there to scramble the letters in the word MISSISSIPPI?

I P S I I M S I P S S

11 possible locations for 2 P's

9 locations left for 4 I's

5 locations left for 4 S's

1 location left for 1 M

$$\binom{11}{2}\ \binom{9}{4}\ \binom{5}{4}\ \binom{1}{1}$$

choices to place P's · choices to place I's · choices to place S's · choices to place M

$$\frac{11!}{2!\,\not{9}!} \times \frac{\not{9}!}{4!\,\not{5}!} \times \frac{\not{5}!}{4!\,\not{1}!} \times \frac{\not{1}!}{1!\,0!}$$

$$\frac{11!}{2!\,4!\,4!\,1!}$$ ways to scramble MISSISSIPPI

permutations with repetitions

The general statement of the principle:

The number of distinct sequences with n_1 1's, n_2 2's, ..., n_k k's, where $n = n_1 + n_2 + ... + n_k$ is

$$\frac{n!}{n_1! n_2! \cdots n_k!}$$

question

How many poker hands (five cards) can be dealt from a deck of 52 cards?

How is this different than r-permutations?

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How many poker hands (five cards) can be dealt from a deck of 52 cards?

How is this different than r-permutations?

In an r-permutation we cared about order. In this case we don't

combinations

An r-combination of a set is a subset of size r

The number of r-combinations out of a set with n elements is denoted as C(n,r) or $\binom{n}{r}$

- {1,3,4} is a 3-combination of {1,2,3,4}
- How many 2-combinations of {a,b,c,d}?

Unordered versus ordered selections

Two ordered selections are the same if

- the elements chosen are the same
- the elements chosen are in the same order.

Ordered selections: r-permutations.

Two unordered selections are the same if

- the elements chosen are the same
  (regardless of the order in which the elements are chosen)

Unordered selections: r-combinations.

Permutations or combinations?

Determine if the situation represents a permutation or a combination:

- In how many ways can three student-council members be elected from five candidates?
- In how many ways can three student-council members be elected from five candidates to fill the positions of president, vice-president and treasurer
- A DJ will play three songs out of 10 requests.

relationship between P(n,r) and C(n,r)

Constructing an r-permutation from a set of n elements can be thought as a 2-step process:

Step 1: Choose a subset of r elements;

Step 2: Choose an ordering of the r-element subset.

Step 1 can be done in C(n,r) different ways.

Step 2 can be done in r! different ways.

Based on the multiplication rule, $P(n,r) = C(n,r) \cdot r!$

Thus

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r! \cdot (n-r)!}$$

r-combinations

How many r-combinations?

$$C(n, r) = \frac{n!}{r!(n-r)!}$$

Note that C(n, 0) = 1

Note that C(n,r) = C(n,n-r)

Example: How many poker hands (five cards) can be dealt from a deck of 52 cards?

C(52,5) = 52! / (5!47!)

r-combinations

How many r-combinations?

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

Note that $C(n, 0) = 1$

$C(n,r)$ satisfies:

$$C(n,r) = C(n,n-r)$$

- We can see that easily without using the formula

combinations or permutations?

How many bit strings of length n contain exactly r ones?

P(n,r) or C(n,r)?

Example

The faculty in biology and computer science want to develop a program in computational biology. A committee of 4 composed of two biologists and two computer scientists is tasked with doing this. How many such committees can be assembled out of 20 CS faculty and 30 biology faculty?

Example

A coin is flipped 10 times, producing either heads or tails. How many possible outcomes

- are there in total?
- contain exactly two heads?
- contain at least three heads?

Example

How many permutations of the letters ABCDEFGH contain the string ABC?

Example

How many 10 character (digits and lowercase/uppercase letters) passwords are possible if

characters cannot be repeated?

a 62^{10} b $C(62, 10)$ c $P(62, 10)$

b) characters can be repeated?

Enumerating Combinations(5,3)

012	123	234
013	124	
014	134	
023		
024		
034		

what is the largest digit we can place in the first position? Can you generalize that for C(n,k)?

How do we do it?

place digits d from lo to hi in position p

then recursively place digits in position p+1

lo: previously placed digit +1

hi: n-k for pos 0, n-k+1 for pos 1, n-k+p for pos p

Now write the code for generating combinations (PA4)