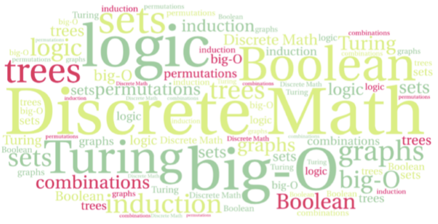
CS 220: Discrete Structures and their Applications

Measuring algorithm running time using big O analysis



Colorado State University

measuring algorithm running time

We have two algorithms: `alg1` and `alg2` that solve the same problem, and you want fast running time.

How do we choose between the algorithms?

Measuring the running time of algorithms

Possible solution:

Implement the two algorithms and compare their running times

Issues with this approach:

- How are the algorithms coded? We want to compare the algorithms, not the implementations.
- What computer should we use? Results may be sensitive to this choice.
- What data should we use?

Measuring the running time of algorithms

Objective: analyze algorithms independently of specific implementations, hardware, or data

Observation: An algorithm's execution time is related to the number of operations it requires

Solution: count the number of steps, i.e. constant time, operations the algorithm will perform for an input of given size

Example: copying an array with n elements requires …. operations.

example: linear search

```
def linear_search(array, value):
    for i in range(len(array)) :
        if array[i] == value :
            return i
    return -1
```

What is the maximum number of steps linear search takes for an array of size n?

example: binary search

```
def binary_search(array, value, lo, hi):
    # precondition: array is sorted
    # postcondition: if value in array[lo...hi] return its position
    # else return -1
    if (lo>hi) :
            r = -1
    else :
        mid = (lo+hi)/2
        if (array[mid]==value):
            r = mid
        elif array[mid]>value :
                r = binary_search(array, value, lo, mid-1)
        else :
            r = binary_search(array, value, mid+1, hi)
    return r
```

time complexity

The time complexity of an algorithm is defined by a function $f: \mathbf{N} \rightarrow \mathbf{N}$ such that $f(n)$ is the maximum number of atomic operations performed by the algorithm on any input of size n.

growth rates

Algorithm A requires $n^2 / 2$ operations to solve a problem of size n

Algorithm B requires $5n+10$ operations to solve a problem of size n

Which one would you choose?

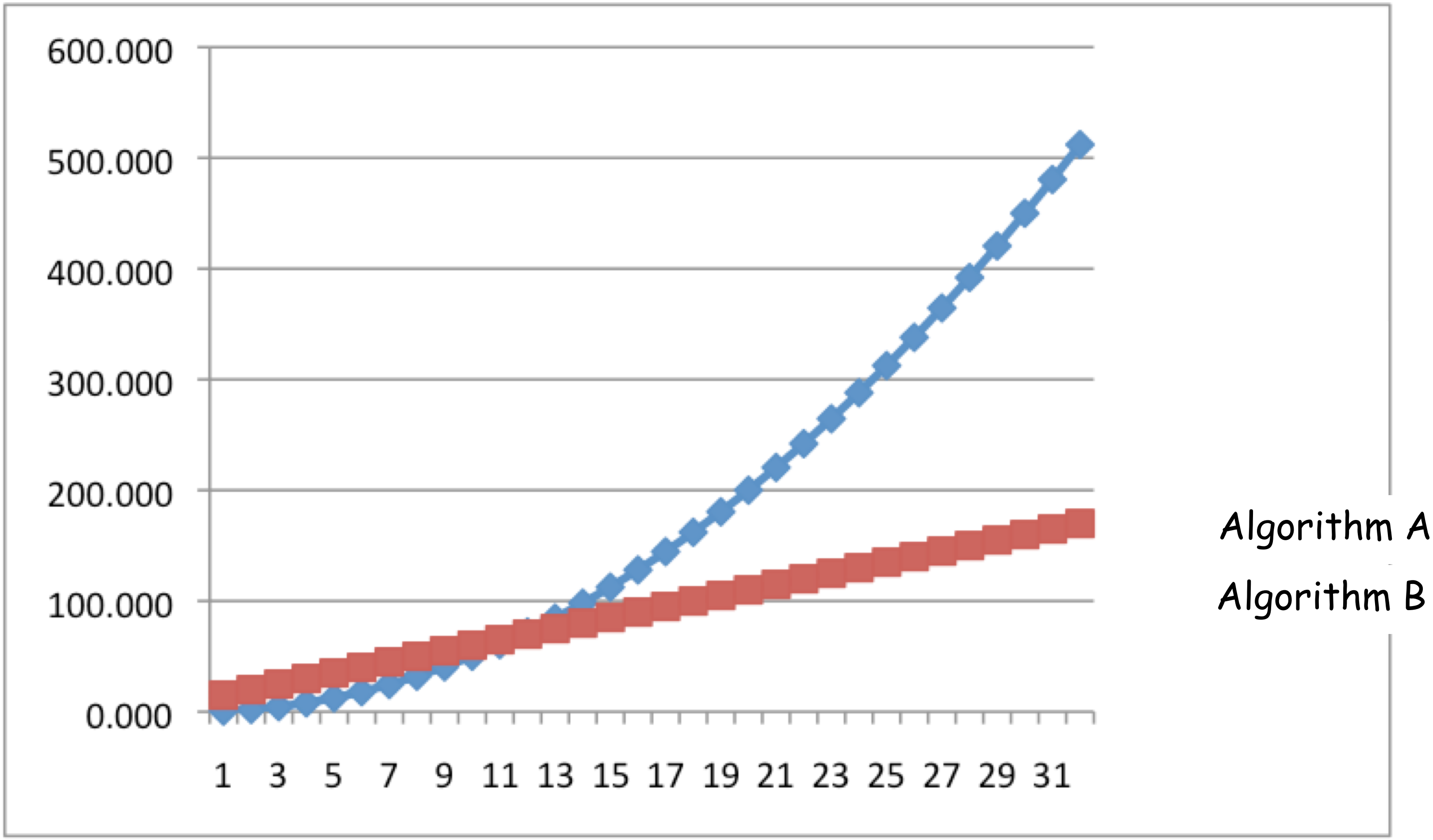
growth rates

When we increase the size of input n, how does the execution time grow for these algorithms?

n	1	2	3	4	5	6	7	8
$n^2 / 2$	.5	2	4.5	8	12.5	18	24.5	32
5n+10	15	20	25	30	35	40	45	50

n	50	100	1,000	10,000	100,000
$n^2 / 2$	1250	5,000	500,000	50,000,000	5,000,000,000
5n+10	260	510	5,010	50,010	500,010

growth rates



growth rates

Algorithm A requires $n^2 / 2$ operations to solve a problem of size n

Algorithm B requires $5n+10$ operations to solve a problem of size n

For large enough problem size algorithm B is more efficient

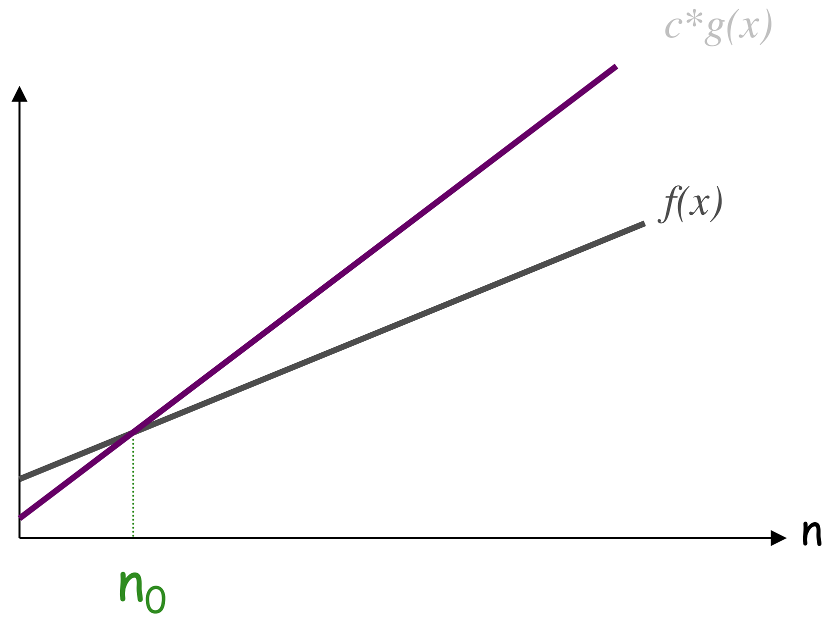
We focus on the growth rate:

- Algorithm A requires time proportional to n^2
- Algorithm B requires time proportional to n

Order of magnitude analysis

Big O: A function $f(n)$ is $O(g(n))$ if there are two positive constants, c and n_0, such that

$$f(n) \leq c * g(n) \qquad \forall n > n_0$$



Order of magnitude analysis

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Focus is on the shape of the function

- Ignore the multiplicative constant

Focus is on large x

- n_0 allows us to ignore behavior for small x

Order of magnitude analysis

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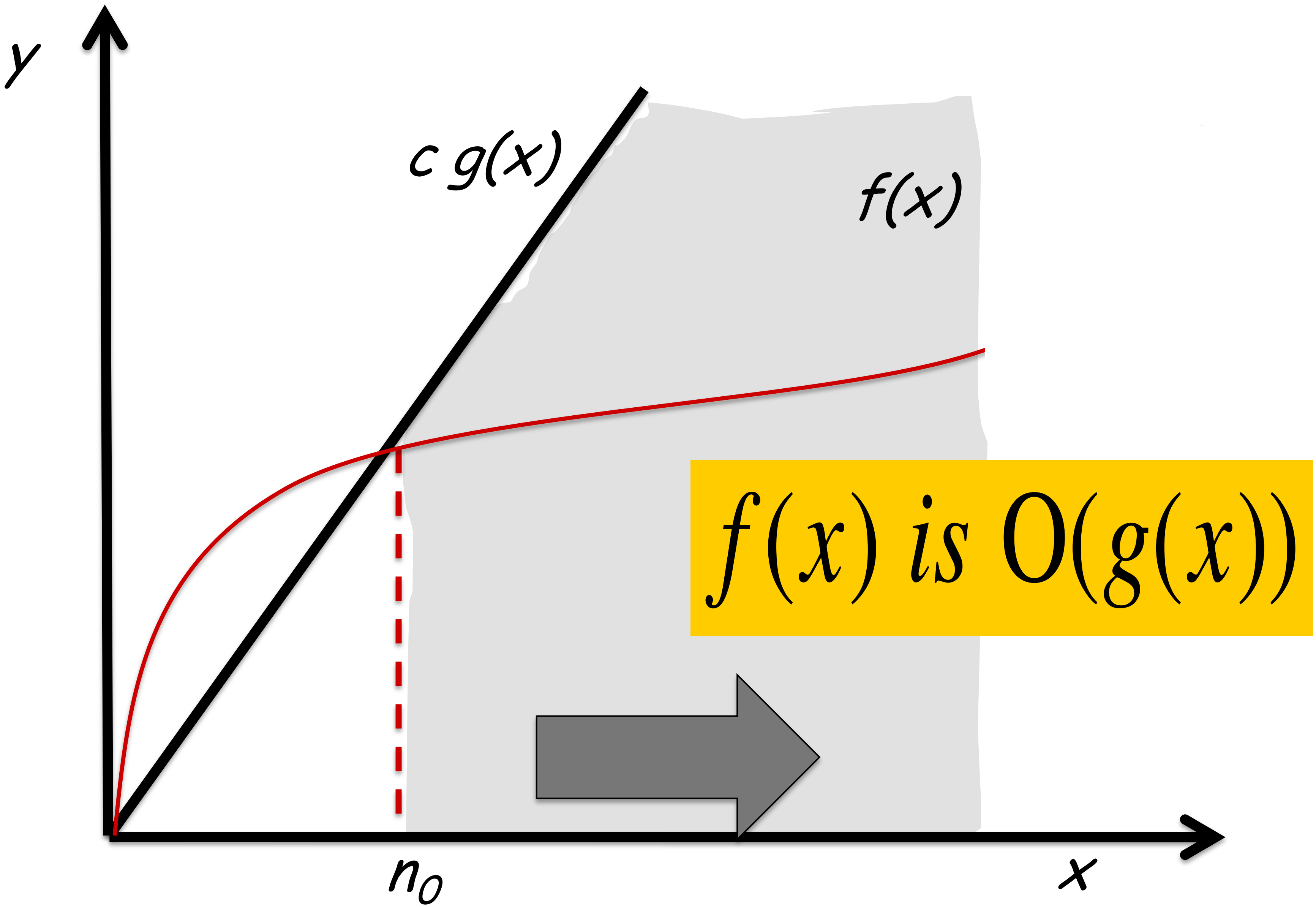
Focus is on the shape of the function

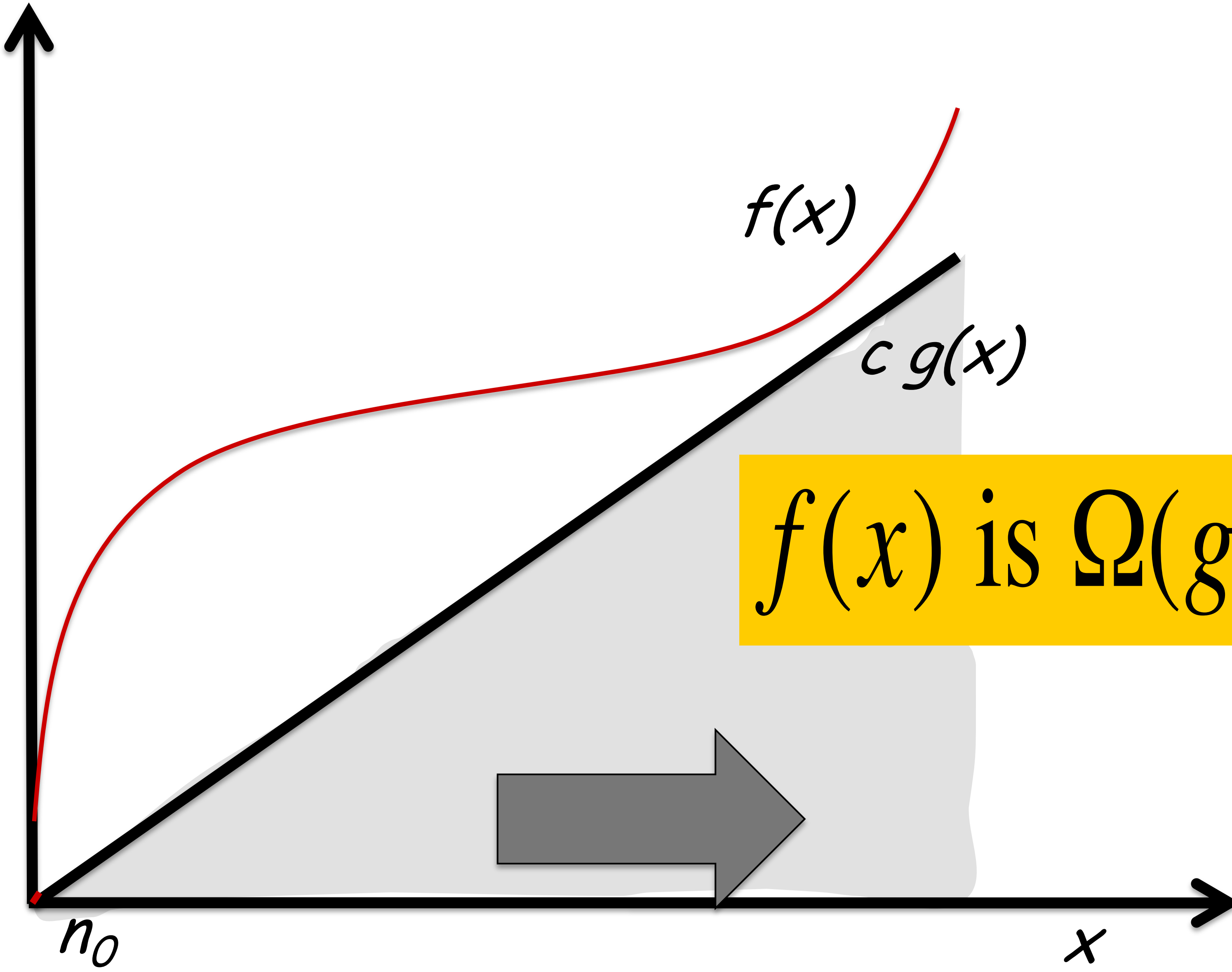
- Ignore the multiplicative constant

Focus is on large x

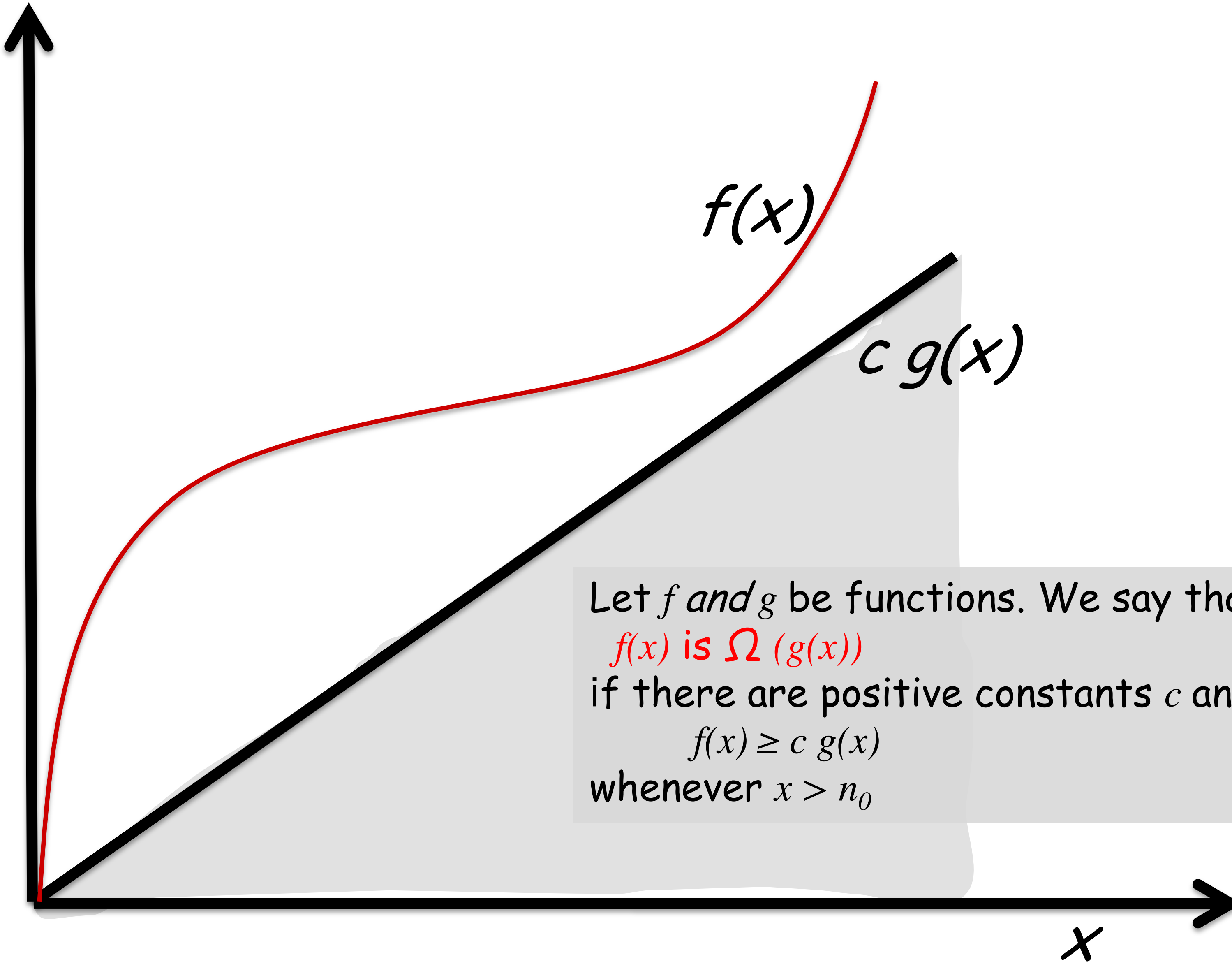
- n_0 allows us to ignore behavior for small x

c and n_0 are witnesses to the relationship that $f(x)$ *is* $O(g(x))$


y
c g(x)
f(x)
f(x) is O(g(x))
n0
x



$$f(x) \text{ is } \Omega(g(x))$$



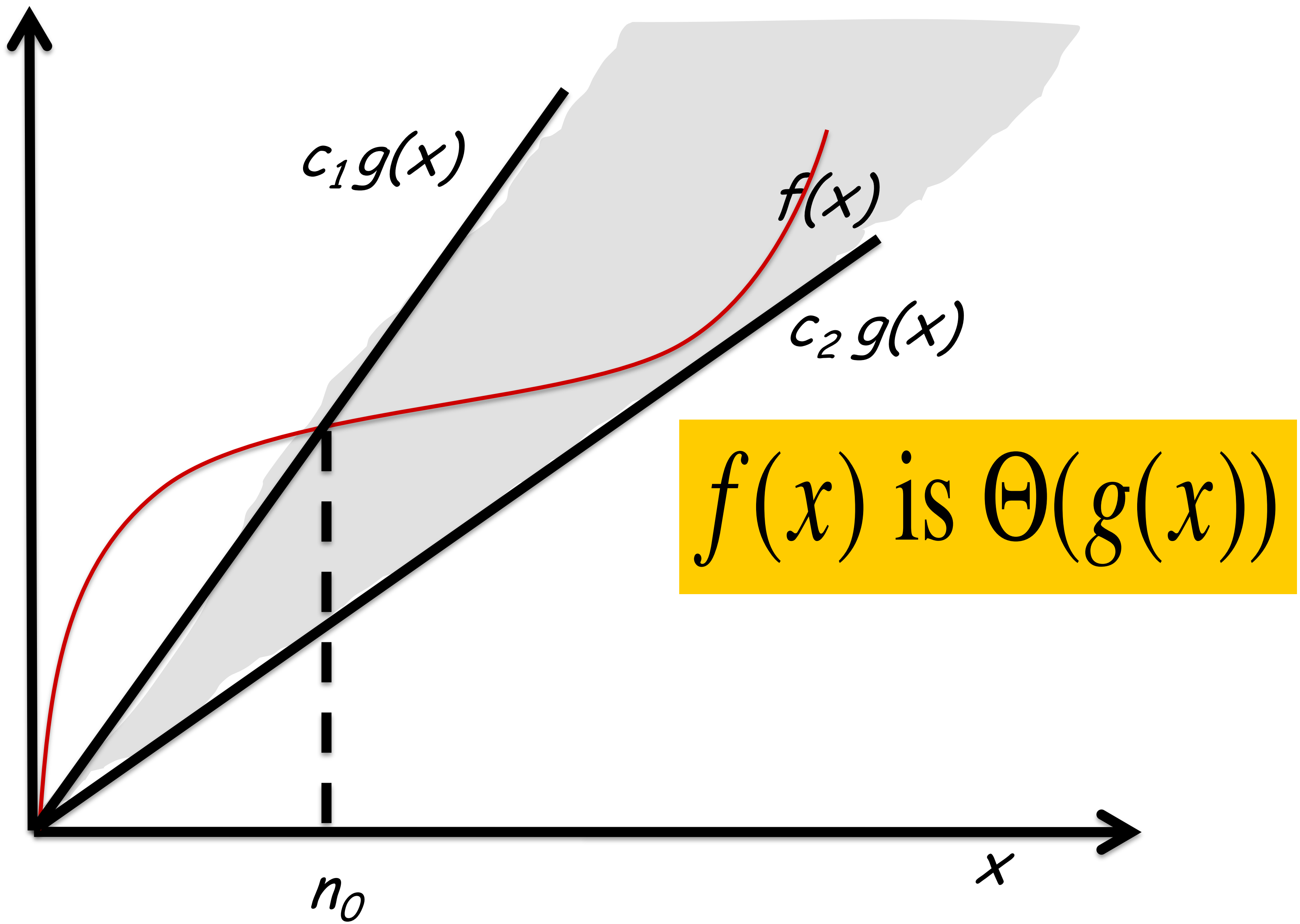
Let f *and* g be functions. We say that

$f(x)$ **is Ω** $(g(x))$

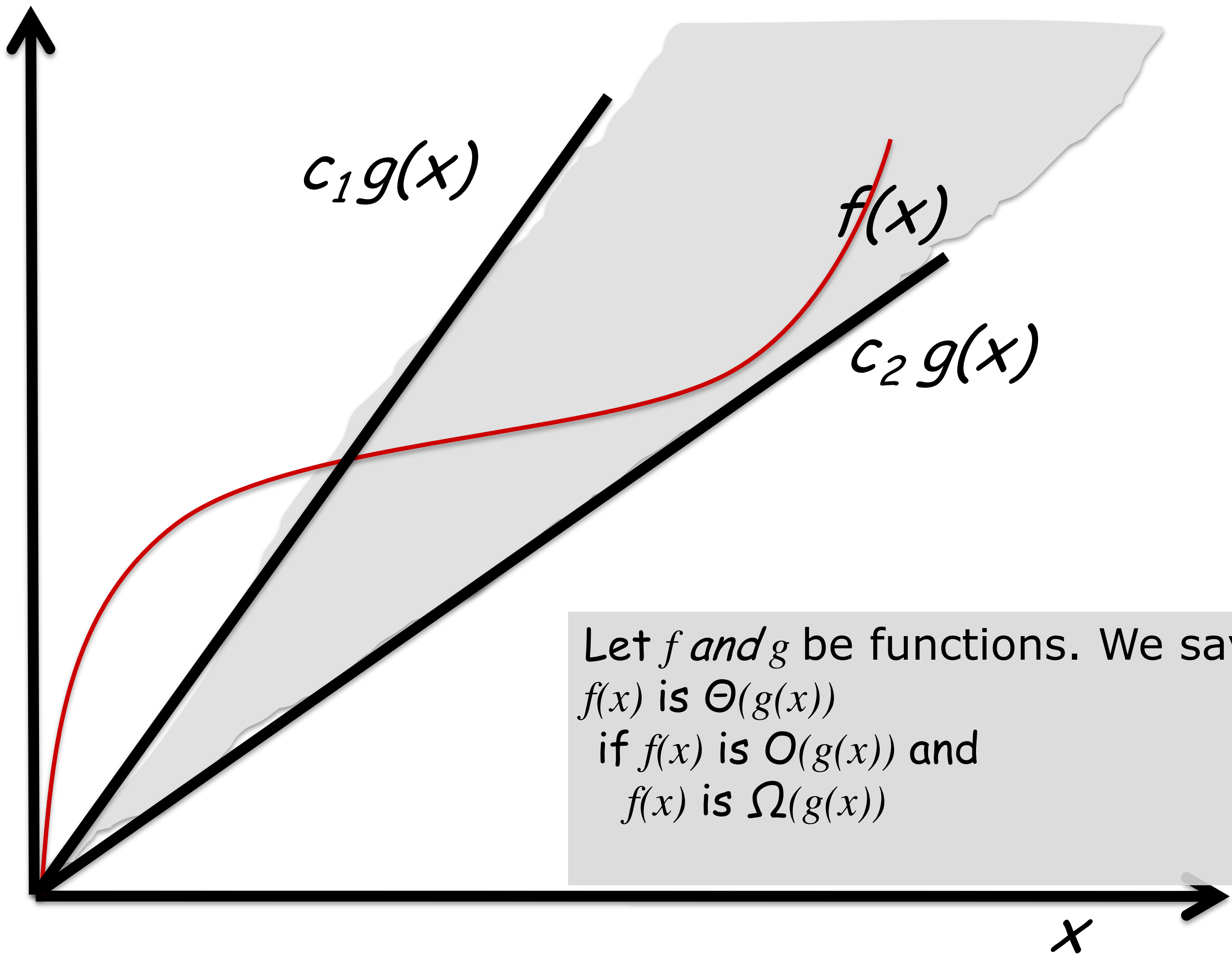
if there are positive constants c and n_0 s.t,

$$f(x) \geq c\ g(x)$$

whenever $x > n_0$



c1g(x)
f(x)
c2 g(x)
f(x) is Θ(g(x))
n0
x



Let f *and* g be functions. We say that
$f(x)$ is $\Theta(g(x))$
if $f(x)$ is $O(g(x))$ and
$f(x)$ is $\Omega(g(x))$

Question

$f(n) = n^2+3n$

Is $f(n)$ $O(n^2)$

why?

Question

$f(n) = n + \log n$

Is $f(n)$ $O(n)$?

why?

Question

$f(n) = n \log n + 2n$

Is $f(n)$ $O(n)$?

why?

Question

$f(n) = n \log n + 2n$

Is $f(n)$ $O(n \log n)$?

why?

worst/average case analysis

Worst case

- just how bad can it *get*: the maximum number of steps
- our focus in this course

Average case

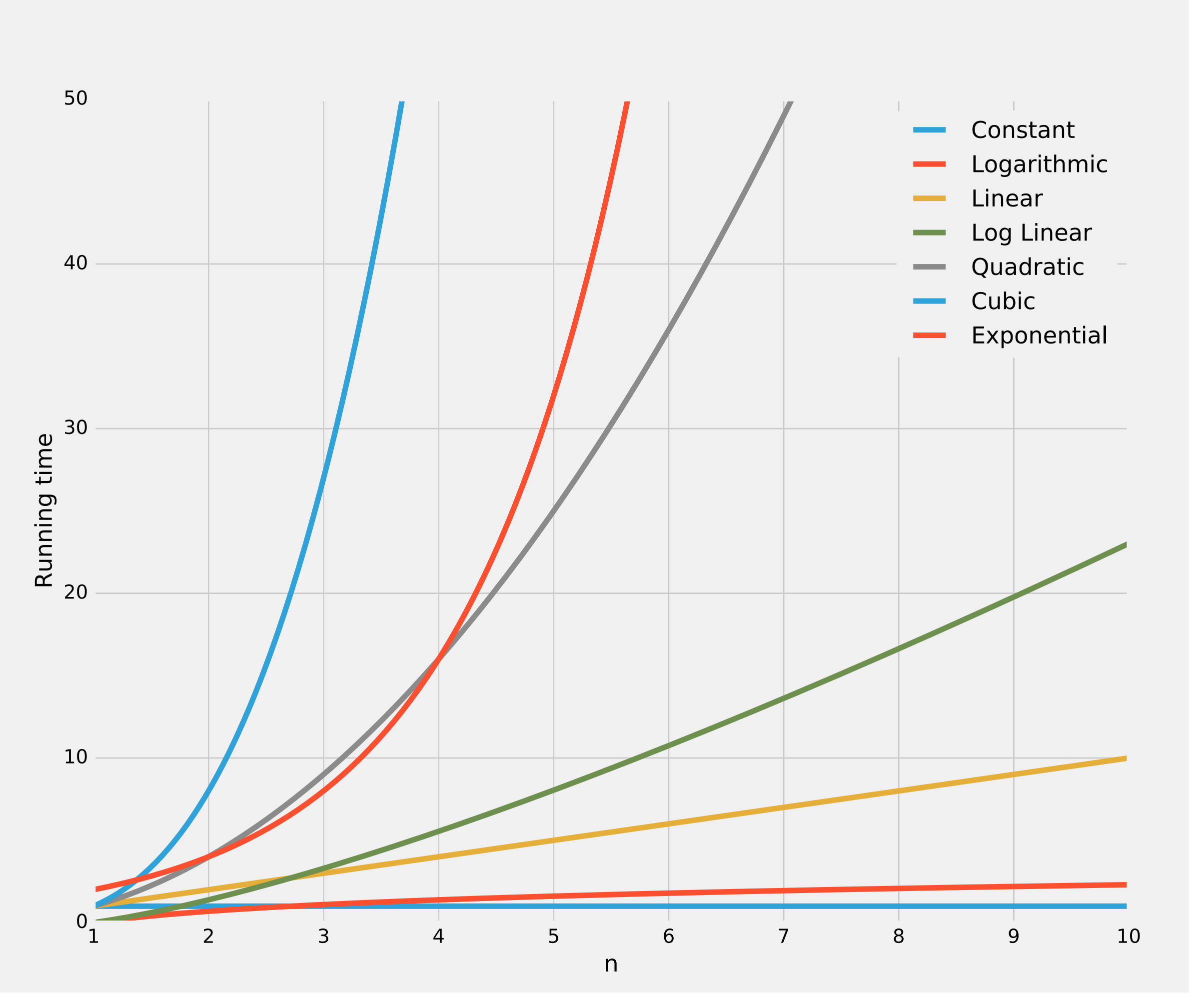
- number of steps expected "usually"
- In this course we will hand wave when it comes to average case

Best case

- The smallest number of steps

Example: searching for an item in an unsorted array

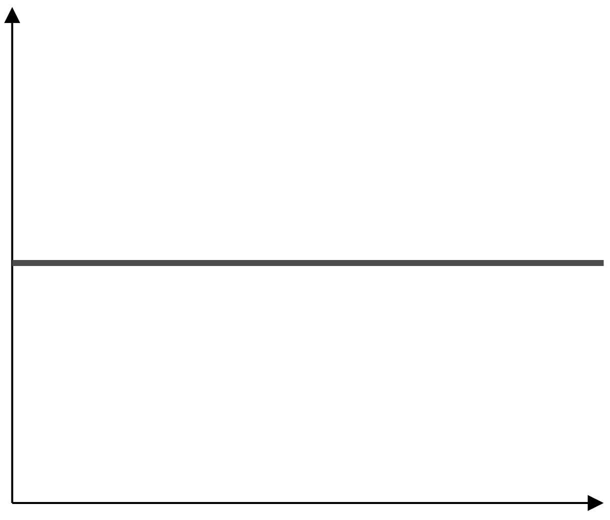
common running times



Careful, this graph is misleading! Why? Small values of n.
Make a table for n^3 and 2^n (n=2,4,8,16,32)

common shapes: constant

$O(1)$



Examples:
Any integer/double arithmetic/logic operation
Accessing a variable or an element in an array

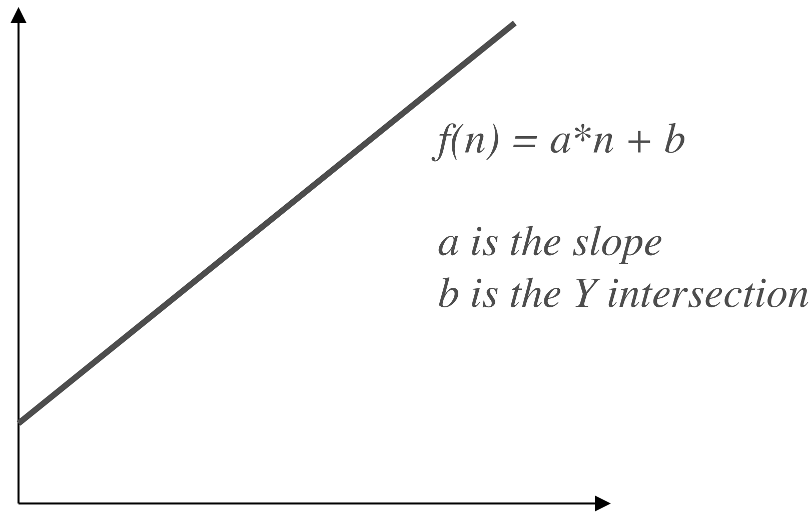
Questions

Which is an example of constant time operations?

A. An integer/double arithmetic operation
B. Accessing an element in an array
C. Determining if a number is even or odd
D. Sorting an array
E. Finding a value in a sorted array

Common Shapes: Linear

O(n)



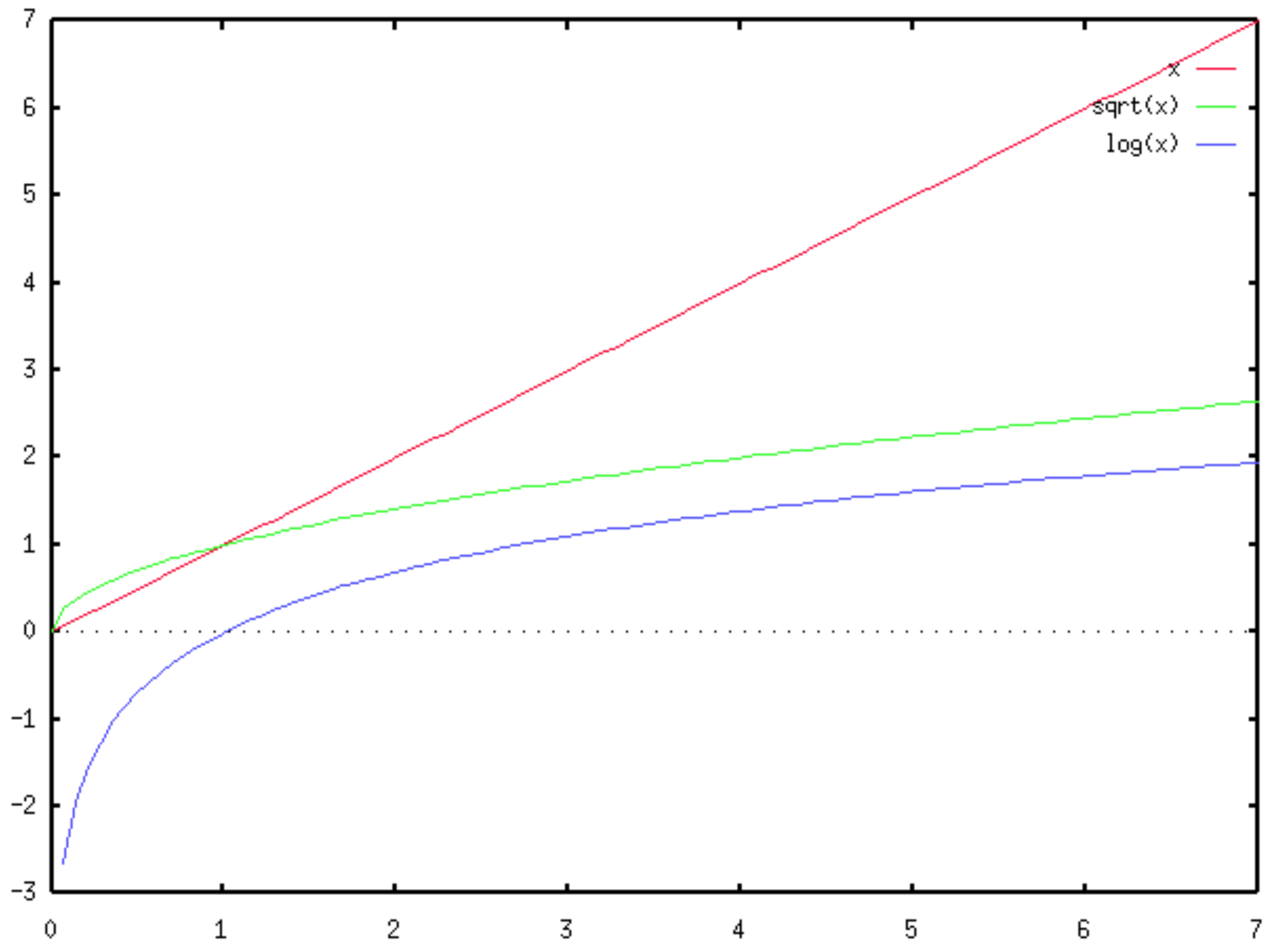
Are all linear functions the same O ?

question

Which are examples of linear time operations?

A. Summing n numbers
B. adding an element in a linked list
C. binary search
D. Accessing A[i] in list A.

Other Shapes: Sublinear



common shapes: logarithm

$log_b n$ is the number x such that $b^x = n$

$2^3 = 8$ $\quad$ $\log_2 8 = 3$

$2^4 = 16$ $\quad$ $\log_2 16 = 4$

$log_b n$: (# of digits to represent n in base b) – 1

We usually work with base 2

$\log_2 n$: number of times you can divide n by 2 until you get to 1

$\log_2 n$ algorithms often break a problem in 2 halves and then solve 1 half

The logarithm is a **very** slow-growing function

Logarithms (cont.)

Properties of logarithms

- $log(x\ y) = log\ x + log\ y$
- $log(x^a) = a\ log\ x$
- $log_a n = log_b n\ /\ log_b a$

  notice that $log_b a$ *is a constant so*

  $\boldsymbol{log_a n = O(log_b n)}$ ***for any a and b***

logarithm is a **very** slow-growing function

Guessing game

I have a number between 0 and 63

How many (Y/N) questions do you need to find it?

is it >= 32 N

is it >= 16 Y

is it >= 24 N

is it >= 20 N

is it >= 18 Y

is it >= 19 Y

What's the number?

Guessing game

I have a number between 0 and 63

How many questions do you need to find it?

is it >= 32	N	0
is it >= 16	Y	1
is it >= 24	N	0
is it >= 20	N	0
is it >= 18	Y	1
is it >= 19	Y	1

What's the number? 19 (010011 in binary)

O(log n) in algorithms

O(log n) occurs in divide and conquer algorithms, when the problem size gets chopped in half (third, quarter,...) every step

(About) how many times do you need to divide

1,000 by 2 to get to 1 ?

1,000,000 ?

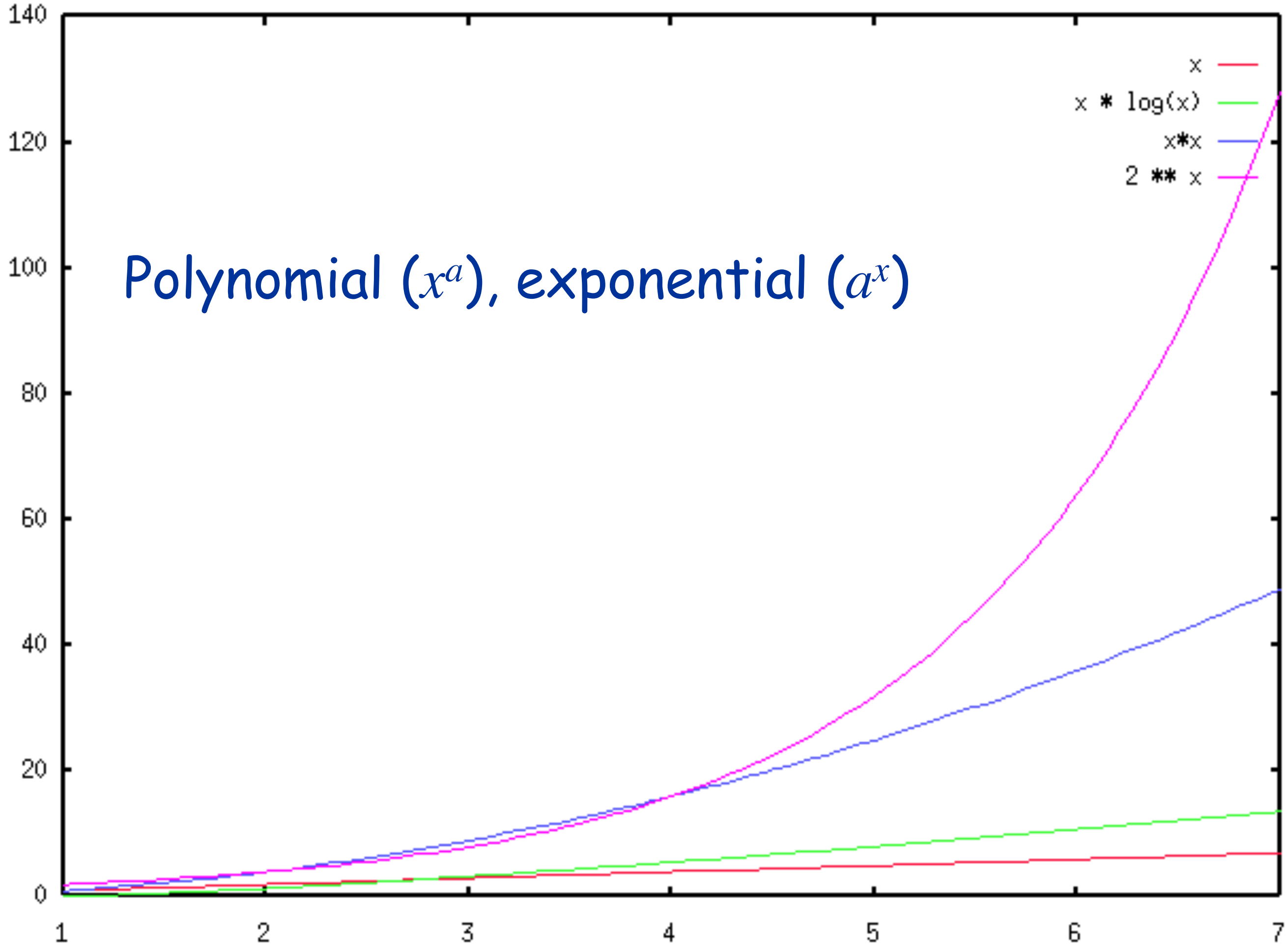
1,000,000,000 ?

Question

Which is an example of a log time operation?

A. Determining max value in an unsorted array
B. Pushing an element onto a stack
C. Binary search in a sorted array
D. Sorting an array

Other Shapes: Superlinear



Polynomial (x^a), exponential (a^x)

quadratic

$O(n^2)$:

```
for i in range(n) :
    for j in range(n) :
      …
```

n times

n times

question

Give a Big O bound for the following function.

$$f(n) = (3n^2 + 8)(n + 1)$$

(a) $O(n)$

(b) $O(n^3)$

(c) $O(n^2)$

(d) $O(1)$

Is $f(n) = O(n^4)$?

What is the BEST (smallest) big O bound for $f(n)$?

Big-O for Polynomials

Theorem: Let

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$$

where $a_n, a_{n-1} ..., a_1, a_0$ are real numbers.

Then $f(x)$ is $O(x^n)$

Example: $x^2 + 5x$ is $O(x^2)$

Are all quadratic functions the same O? All cubic?

combinations of functions

Additive Theorem:

Suppose that $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$.
Then $(f_1 + f_2)(x)$ is $O(\max(|g_1(x)|,|g_2(x)|))$.

Multiplicative Theorem:

Suppose that $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$.
Then $(f_1 f_2)(x)$ is $O(g_1(x)g_2(x))$.

practical analysis

Sequential

- Big-O bound: steepest growth dominates
- Example: copying of array, followed by binary search
  - n + log(n)   O(?)

Embedded code

- Big-O bound multiplicative
- Example: a for loop with n iterations and a body taking O(log n)   O(?)

dependent loops

```
....
for (i = 0; i < n; i++) {
        for (j = 0; j < i; j++){
            ...
        }
}
...
```

i = 0: inner-loop iters = 0

i = 1: inner-loop iters = 1

.
.
.

i = n-1: inner-loop iters = n-1

Total = 0 + 1 + 2 + ... + (n-1)

$f(n) = n*(n-1)/2$

$O(n^2)$

Loop Example

```
public int f7(int n){
        int s = n;
        int c = 0;
        while(s>1){
                s/=2;
                for(int i=0;i<n;i++)
                        for(int j=0;j<=i;j++)
                                c++;
        }
        return c;
}
```

How many outer (while) iterations?

How many inner
for i
    for j
iterations?

Big O complexity?

Loop Example

```
public int f7(int n){
        int s = n;
        int c = 0;
        while(s>1){
                s/=2;
                for(int i=0;i<s;i++)
                    c++;
        }
        return c;
}
```

How many outer (while) iterations?

How many inner for i per s? iterations?

Big O complexity?

recursion

Number of operations depends on :

- number of calls
- work done in each call

Examples:

- factorial: how many recursive calls?
- binary search?
- merge sort?
- Fibonacci? (hint: draw the call tree)

```
def h(n):
    if n==1: return 1
    else: return h(n-1) + h(n-1)

def f(n):
    if n<2: return 1
    else: return f(n-1) + f(n-2)
```

Practical Analysis - Recursion

Number of operations depends on :

- number of calls
- work done in each call

Examples:

- factorial: how many recursive calls?
- binary search?

We will devote more time to analyzing recursive algorithms later in the course.

Example Recursive Code

```
public int divCo(int n){
        if(n<=1)
                return 1;
        else
                return 1 + divCo(n-1) + divCo(n-1);
}
```

How many recursive calls?
hint: draw the call tree

Big O complexity?

How much work per call?
What is the role of “return 1” and return 1+…” ?

final comments

- Order-of-magnitude analysis focuses on large problems
- If the problem size is always small, you can probably ignore an algorithm's efficiency
- Weigh the trade-offs between an algorithm's time requirements and its memory requirements, expense of programming/maintenance…