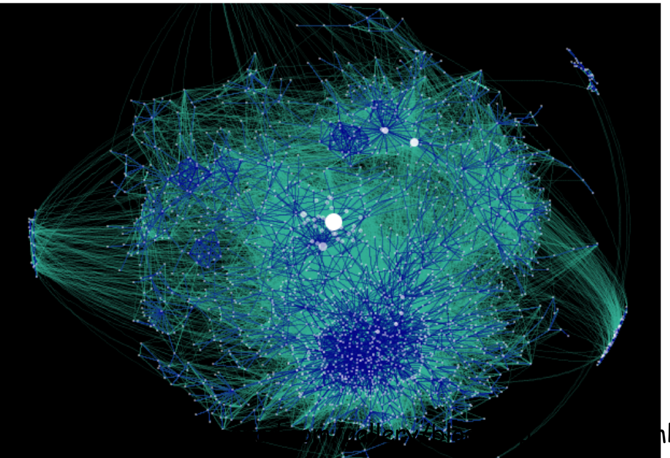
CS 220: Discrete Structures and their Applications

relations and directed graphs; transitive closure

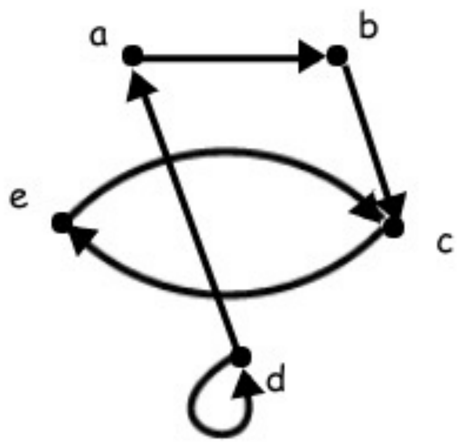
zybooks 9.3-9.6



binary relations on a set

A binary relation on a set A is a subset of $A \times A$.

Graphical representation of a binary relation on a set:



self loop

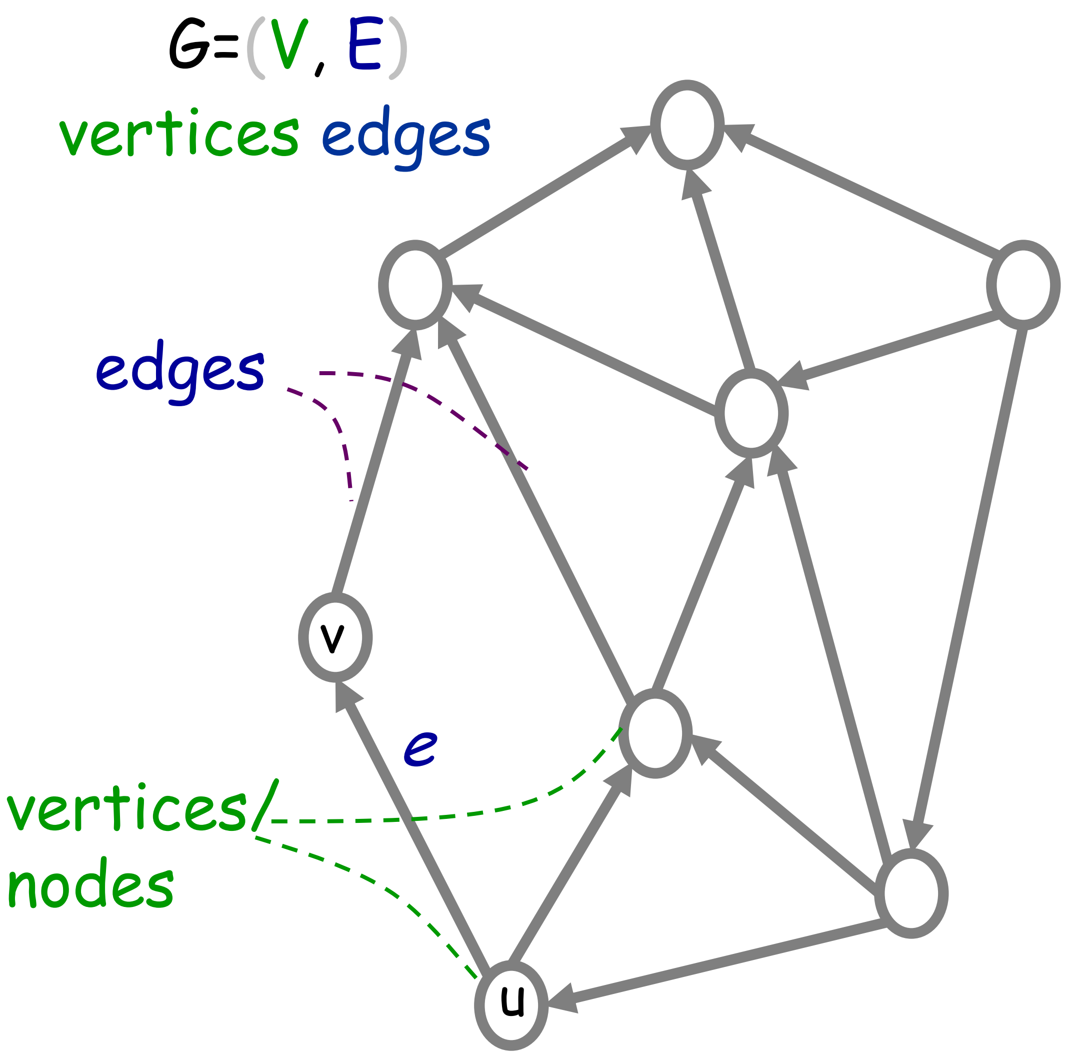
$A = \{a, b, c, d, e\}$

$R \subseteq A \times A$

$R = \{(a, b)(b, c)(e, c)(c, e)(d, a)(d, d)\}$

This special case of a binary relation is also called a directed graph

directed graphs



Edge (u, v) goes from vertex u to vertex v.

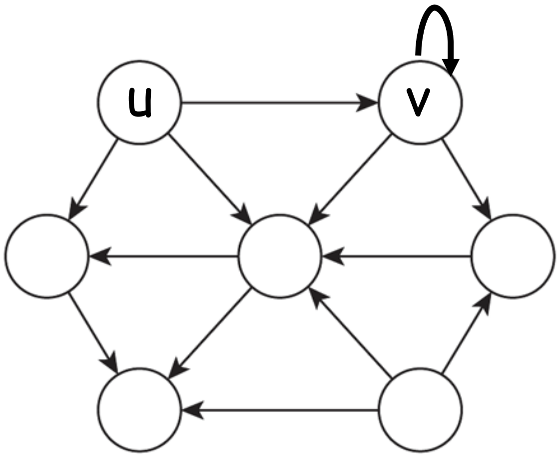
in-degree of a vertex: the number of edges pointing into it.
out-degree of a vertex: the number of edges pointing out of it.

terminology

A directed graph (or digraph) is a pair (V, E).

V is a set of vertices, and E, a set of directed edges, is a subset of V × V.

The vertex u is the **tail** of the edge (u, v) and vertex v is the **head**. If the head and the tail of an edge are the same vertex, the edge is called a **self-loop**.



Example: The web. What are the vertices/edges?

matrices

An $n \times m$ matrix over a set S is an array of elements from S with n rows and m columns.

The entry in row i and column j is denoted by $A_{i,j}$.

A matrix is called a square matrix if the number of rows is equal to the number of columns.

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

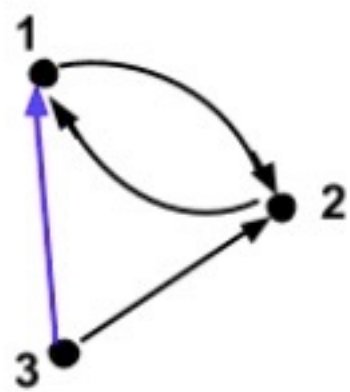
2 x 2 matrix over {0, 1}

Is the adjacency matrix associated with a graph square?

adjacency matrix

A directed graph G with n vertices can be represented by an $n \times n$ matrix over the set $\{0, 1\}$ called the adjacency matrix for G.

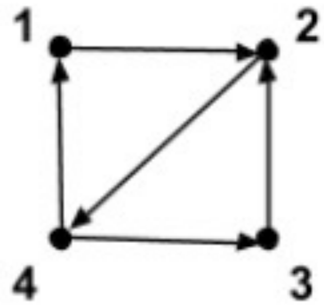
If A is the adjacency matrix for a graph G, then $A_{i,j} = 1$ if there is an edge from vertex i to vertex j in G. Otherwise, $A_{i,j} = 0$.



$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

adjacency matrix

What are the missing values in the following adjacency matrix?



$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ ? & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & ? & 0 \end{pmatrix}$$

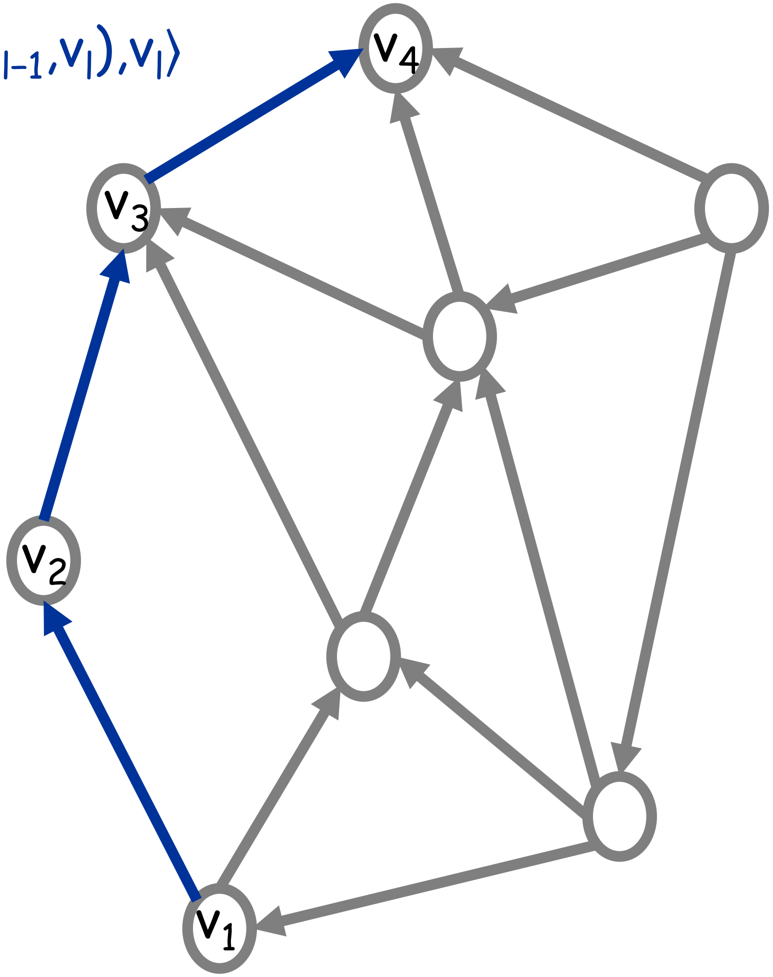
$A_{2,1} = ?$

$A_{4,3} = ?$

walks

A walk from v_0 to v_l in a directed graph G is a sequence of alternating vertices and edges that starts and ends with a vertex:

$\langle v_0,(v_0,v_1),v_1,(v_1,v_2),v_2,\ldots,v_{l-1},(v_{l-1},v_l),v_l\rangle$



walks

A walk from v_0 to v_l in a directed graph G is a sequence of alternating vertices and edges that starts and ends with a vertex:

$\langle v_0,(v_0,v_1),v_1,(v_1,v_2),v_2,\ldots,v_{l-1},(v_{l-1},v_l),v_l\rangle$

A walk can also be denoted by the sequence of vertices:

$\langle v_0,v_1,\ldots,v_l\rangle$.

The sequence of vertices is a walk only if $(v_{i-1}, v_i) \in E$

for $i = 1, 2,\ldots,l$.

The length of a walk is l, the number of edges in the walk.

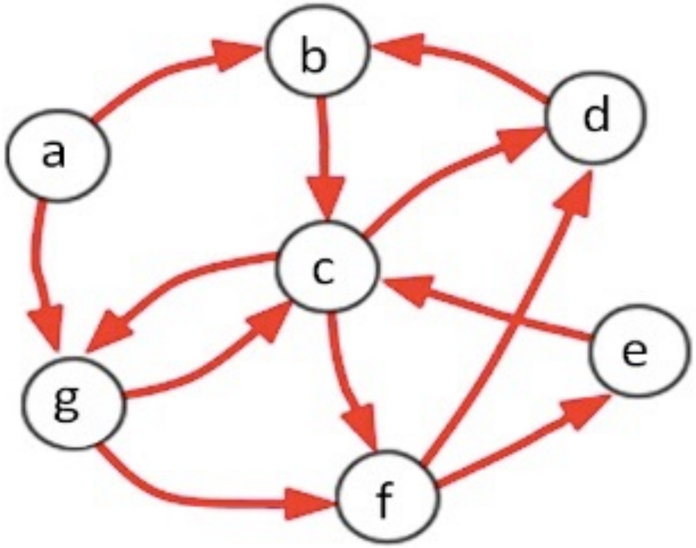
walks, circuits, paths, cycles

A circuit is a walk in which the first vertex is the same as the last vertex.

A sequence of one vertex, denoted <a>, is a circuit of length 0.

A walk is a path if no vertex is repeated in the walk.

A circuit is a cycle if there are no other repeated vertices, except the first and the last.



composite relations

Let R be a relation from A to B, and let S be a relation from B to C. The composite $S \circ R$ of R and S is defined as:

$$S \circ R = \{(a,c) : \exists b \text{ such that } aRb \text{ and } bSc\}$$

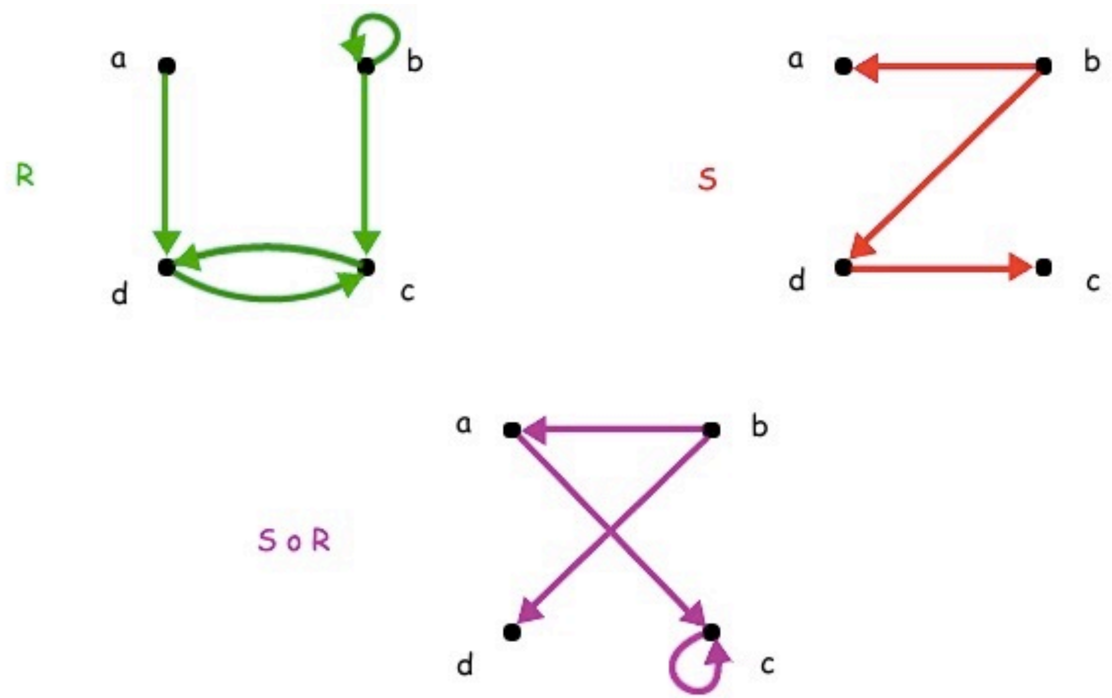
Example: Let R be the relation such that aRb if a is a parent of b. What is the relation $R \circ R$?

composite relations

Let R be a relation from A to B, and let S be a relation from B to C. The composite $S \circ R$ of R and S is defined as:

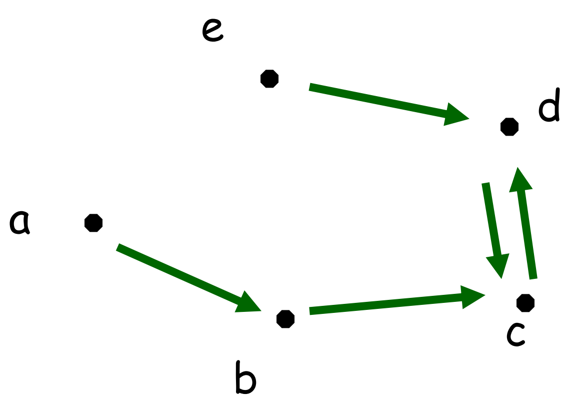
$$S \circ R = \{(a,c) : \exists b \text{ such that } aRb \text{ and } bSc\}$$

Example:

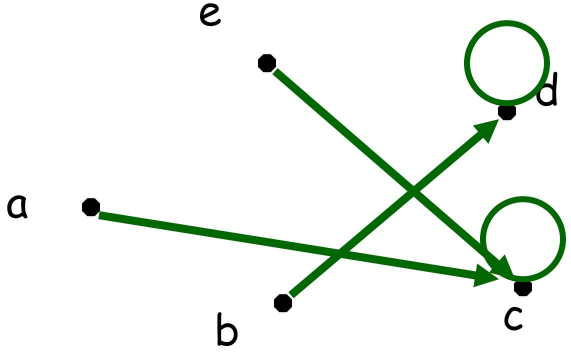


composite relations

Composite relation on a set:



R



R^2

composite relations

The powers R^n of relation R can be defined recursively:

$$R^1 = R \text{ and } R^{n+1} = R^n \circ R$$

The statement of *six-degrees of separation* can be succinctly expressed as aR^6b for all a,b where R is the relation on the set of people such that aRb if a knows b

paths and relations

The edge set E of a directed graph G can be viewed as a relation.

E^k : the relation E composed with itself k times.

G^k : the directed graph whose edge set is E^k.

The Graph Power Theorem: Let G be a directed graph. Let u and v be any two vertices in G. There is an edge from u to v in G^k if and only if there is a walk of length k from u to v in G.

paths and relations

The Graph Power Theorem: Let G be a directed graph. Let u and v be any two vertices in G. There is an edge from u to v in G^k if and only if there is a walk of length k from u to v in G.

Proof by induction on k.

Base case: there is a walk of length 1 iff aRb

Induction step: assume that there is an edge from u to v in G^k if and only if there is a walk of length k from u to v in G.

Let's prove this for k+1:

There is a walk of length k+1 from a to b iff there is a c in A such that there is a walk of length 1 from a to c, i.e. aRc and a path of length k from c to b; by the induction hypothesis this happens iff cR^kb, and by definition of composition iff $aR^{k+1}b$

the transitive closure

The transitive closure of a graph G:

$$G^+ = G^1 \cup G^2 \cup G^3 \cup G^4 \ldots.$$

In the union, there is only one copy of the vertex set and the union is taken over the edge sets of the graphs.

(u, v) is an edge in G^+ if vertex v can be reached from vertex u in G by a walk of any length.

Similarly we can define the transitive closure of a relation R:

$$R^+ = R^1 \cup R^2 \cup R^3 \cup R^4 \ldots.$$

the transitive closure

Examples:

Let R be the relation between states in the US where aRb if a and b share a common border. What is R^+?

What is R^+ for the parent relation?

A = {1, 2, b}. What is the transitive closure for:

- R = {(1, 1), (b, b)}
- S = {(1, 2), (2, b)}
- T = {(2, 1), (b, 2),(1,1)}

the transitive closure

The transitive closure of a graph G:

$$G^+ = G^1 \cup G^2 \cup G^3 \cup G^4 \ldots.$$

If the graph has n vertices:

$$G^+ = G^1 \cup G^2 \cup G^3 \cup \ldots. \cup G^n$$

The same holds for a relation R. Let R be a relation on a finite domain with n elements. Then

$$R^+ = R^1 \cup R^2 \cup R^3 \cup \ldots. \cup R^n$$

the transitive closure

Lemma: Let G be a graph with n vertices. If there is a path from u to v in G, then there is such a path with length not exceeding n.

an algorithm for the transitive closure

Let R be a relation over a set A.

Repeat the following step until no pair is added to R:

- ✓ If there are x, y, z ∈ A such that (x, y) ∈ R, (y, z) ∈ R and (x, z) ∉ R, then add (x, z) to R.