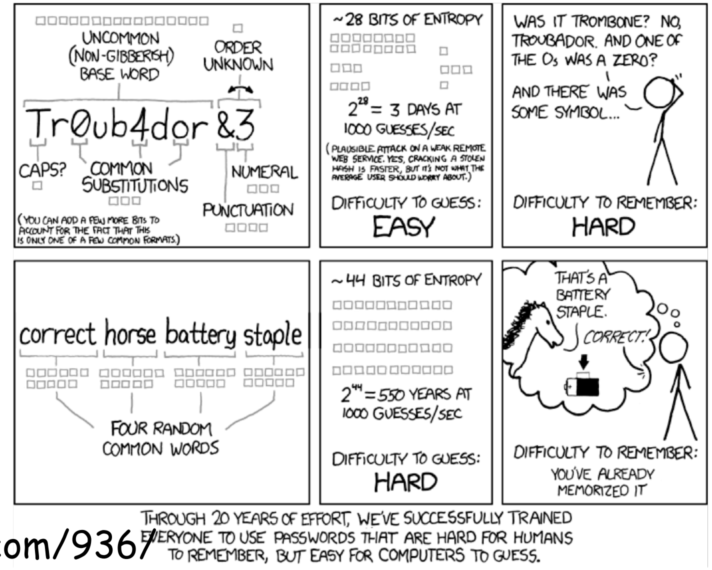
CS 220: Discrete Structures and their Applications

Counting: the sum and product rules

zybooks 7.1 - 7.2



http://www.xkcd.com/936/

Why count

Counting is an important aspect of algorithm design and complexity analysis. We need to count:

- the number of loop iterations to establish the time complexity of our programs
- The number of elements of our arrays / lists /dictionaries to establish the space complexity of our programs

A simple counting problem

You have 6 pairs of pants and 10 shirts. How many different outfits does this give?

Possible answers:

A) 6×10

B) $6 + 10$

enumerating outfits

You have 6 pairs of pants and 10 shirts. How many different outfits does this give?

We can express the set of all outfits as:

$$\{(s, p) \mid s \in \text{shirts and } p \in \text{pants}\}$$

enumerating outfits

You have 6 pairs of pants and 10 shirts. How many different outfits does this give?

How would you write a program that prints out all the possible outfits? (Assume you have an array of shirts and an array of pants).

relation to Cartesian products

The Cartesian product of sets A and B is denoted by A x B and is defined as:

$A \times B = \{ (a,b) \mid a \in A \text{ and } b \in B\}$

The product rule is simply a statement about the cardinality of a Cartesian product:

$| A \times B | = | A | * | B |$

the product rule

The general statement of the product rule:

Let $A_1, A_2, \ldots, A_n$ be finite sets. Then,

$$|A_1 \times A_2 \times \ldots \times A_n| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_n|$$

the product rule

Example: counting strings

Let Σ is a set of characters (i.e. an alphabet);

Σ^n is the set of all strings of length n whose characters come from the set Σ.

Applying the product rule:

$$|\Sigma^n| = |\Sigma \times \Sigma \times \cdots \times \Sigma| = |\Sigma| \cdot |\Sigma| \cdots |\Sigma| = |\Sigma|^n$$

Therefore, there are 2^n binary strings

the product rule

Colorado assigns license plates numbers as three digits followed by three uppercase letters. How many license plate numbers are possible?

the product rule

Colorado assigns license plates numbers as three digits followed by three uppercase letters. How many license plate numbers are possible?

A) $3^{10} \times 3^{26}$

B) $2 \times 3^{10} \times 3^{26}$

C) $10^{3} \times 26^{3}$

more examples

How many bit strings of length 7 are there?

A) 7×6 B) 7^2 C) 2^7

How many functions are there from a set with m elements to a set with n elements?

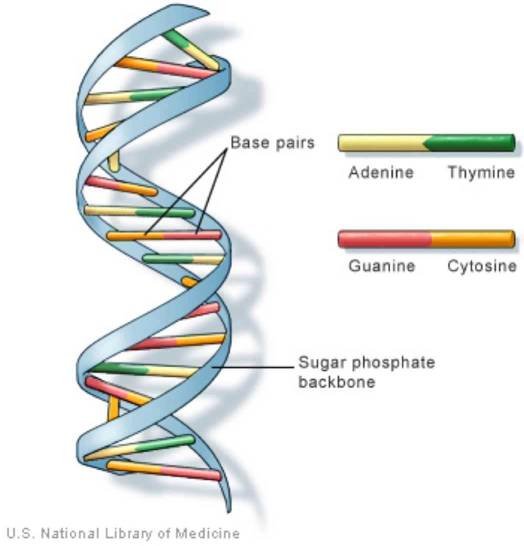
DNA and proteins

DNA is a long chain that contains one of four nucleotides (A,C,G,T). DNA can code for proteins that are chains of amino acids. There are 20 amino acids. How many nucleotides does it take to code for a single amino acid?

A) 2

B) 3

C) 4



another counting problem

Suppose you order a drink, and you can select either a hot drink or a cold drink. The hot drink selections are {coffee, hot cocoa, tea}. The cold drink selections are {milk, orange juice}. What is the total number of choices?

A) $3 + 2$

B) 3×2

C) $3! \times 2!$

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The difference from the product rule: there is a single choice to be made.

the sum rule

The sum rule is also statement about set theory:

If two sets A and B are disjoint then

$$|A \cup B| = |A| + |B|$$

the sum rule

The general form of the sum rule:

Consider n sets, A_1, A_2,…,A_n. If the sets are mutually disjoint ($A_i \cap A_j = \emptyset$ for $i \neq j$), then

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1| + |A_2| + \ldots + |A_n|$$

example

A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 4 possible projects. No project is on more than one list. How many possible projects are there to choose from?

example

The product and sum rule can be combined:

Suppose you need to pick a password that has length 6-8 characters, where each character is an uppercase letter or a digit. How many possible passwords are there?

example

The product and sum rule can be combined:

How many license plates can be made using either two or three uppercase letters followed by two or three digits?

The generalized product rule

Consider the following counting problem:

In a race with 20 runners there is a first place, a second place and a third place trophy. An outcome of the race is defined to be who wins each of the three trophies, i.e. 3 distinct runners. How many outcomes are possible?

The generalized product rule

Choosing sequences of items

Consider a set S of sequences of k items. Suppose there are:

- ✓ n_1 choices for the first item.
- ✓ For every possible choice for the first item, there are n_2 choices for the second item.
- ✓ For every possible choice for the first and second items, there are n_3 choices for the third item.

⋮

- ✓ For every possible choice for the first k-1 items, there are n_k choices for the kth item

Then $|S| = n_1 \cdot n_2 \cdots n_k$.

example

A family of four (2 parents and 2 kids) goes on a hiking trip. The trail is narrow and they must walk single file. How many ways can they walk with a parent in the front and a parent in the rear?

example

Consider the following definitions for sets of characters:

Digits = { 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 }

Letters = { a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z }

Special characters = { *, &, $, # }

Compute the number of passwords that satisfy the following constraints:

(a) Strings of length 6. Characters can be special characters, digits, or letters, with no repeated characters.

(b) Strings of length 6. Characters can be special characters, digits, or letters, with no repeated characters. The first character cannot be a special character.

Making Change

Goal. Given integer coin values, e.g.: {1,5,10,25} compute in how many ways you can pay a certain amount:

Example: 29¢.

How many ways?

25, 1,1,1,1

10,10, 5, 1,1,1,1

10, 5,5, 5, 1,1,1,1

10, 5,5, 1,1,1,1,1, 1,1,1,1

. . .

1,1,1,1,1, 1,1,1,1,1, 1,1,1,1,1, 1,1,1,1,1, 1,1,1,1,1, 1,1,1,1

Making Change

Given a coin set c = {c_0, c_1, ..., c_{n-1}} and an amount M, how many different ways can M be paid?

Recursive solution: d=n-1, given coin value c_d , how many coins can I use ?

e.g., for eg 56 cents I can use 0, 1, or 2 quarters

Base:

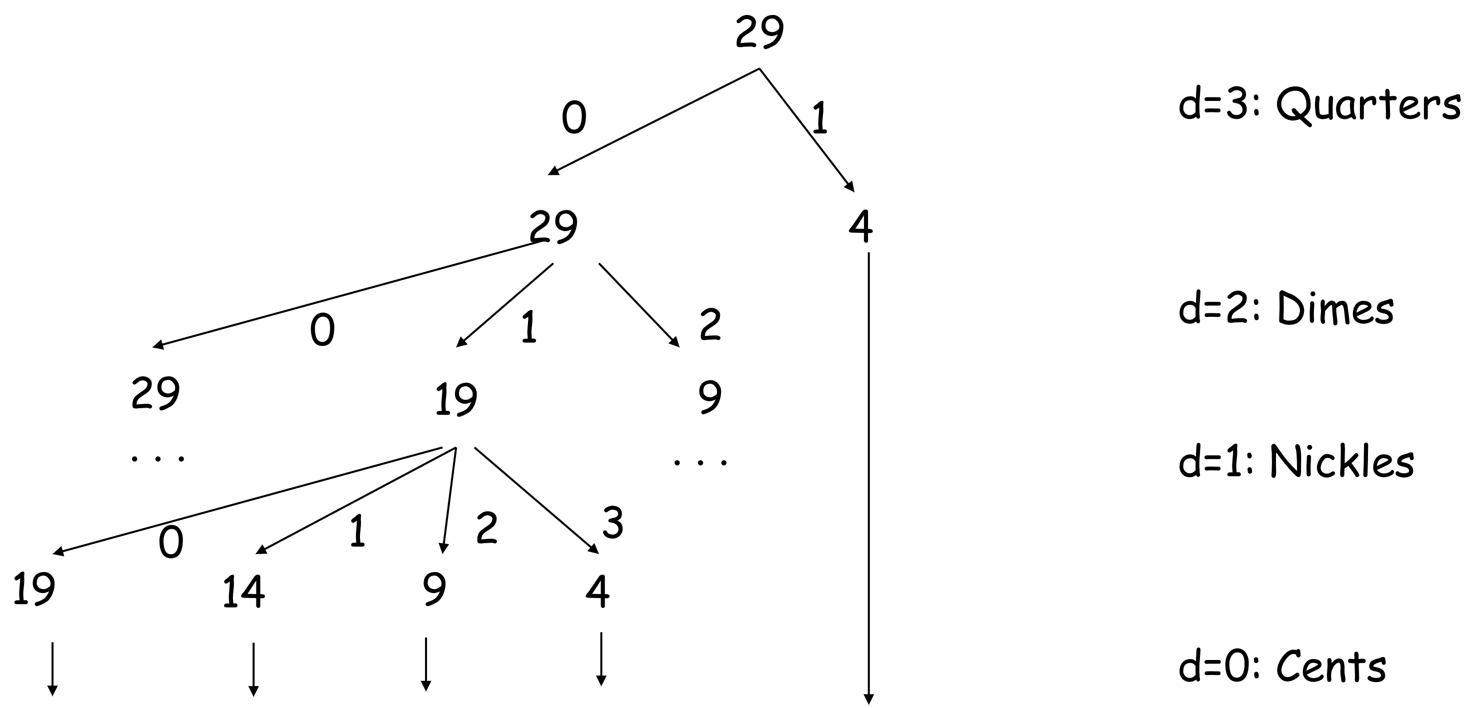
if d == 0, how many ways? (is there always a way ?)

Step:

if d>0, go through all possible uses of coin d

at least how many c_d coins can be used
and which problem then remains to be solved?
...
at most how many c_d coins can be used
and which problem then remains to be solved?

Making Change



PA3

One of your tasks in PA3 is to write a (recursive) function

mkCh()

that counts

the number ways a certain amount of money

can be paid with a coin set {1,5,10,25}

k out of n partitions

Given n distinct elements, we want to group these into k partitions. E.g. n=3 {b,c,d} How many 2 out of 3 partitions are there?

Enumerate . . .

{b} {c,d}
{c} {b,d}
{d} {b,c}

there are 3 2-out-of-3 partitions

Counting the number of k-out-of-n partitions

Now n=4 {a,b,c,d} How many 3-out-of-4 partitions are there?

Typical Divide and Conquer (hence recursive) approach: take element a

There are two possibilities:

1) either a is in its own partition or 2) not

We can apply the sum rule

1) a is in its own partition, then there are 2 more partitions out of three elements {b,c,d}. We already solved how many 2 out of 3 partitions there are:

{b} {c,d}

{c} {b,d}

{d} {b,c}

So for this case we get 3 solutions

{a} {b} {c,d}

{a} {c} {b,d}

{a} {d} {b,c}

Second possibility

2) a is not in its own partition, then is it in 1 of the 3 3-out-of-3 partitions:

{b} {c} {d}

so in case 2) we have 3 possibilities

{a,b} {c} {d}

{b} {a,c} {d}

{b} {c} {a,d}

So in total there are 6 3-out-of-4 partitions

Do it yourself

How many 2-out-of-4 partitions are there?

Do it yourself

How many 2-out-of-4 partitions are there?

{a,b,c,d}

Option 1: a on its own

{a} {b,c,d}

Option 2: a joins one of:

{b} {c,d} → {a,b} {c,d}

{b} {a,c,d}

{c} {b,d} → {a,c} {b,d}

{c} {a,b,d}

{d} {b,c} → {a,d} {b,c}

{d} {a,b,c}

So in total 7 (1 + 2*3) 2_out_of_4 groups

PA3

One of your tasks in PA3 is to write a (recursive) function

partitions(n,k)

that counts

the number of k-out-of-n partitions