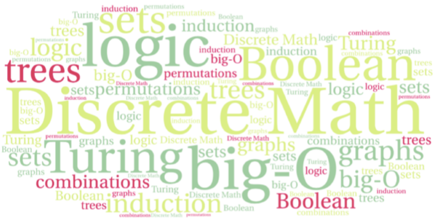
CS 220: Discrete Structures and their Applications

partial orders, DAGs and
n-ary relations



Recap Binary Relations

The relation R is reflexive if for every $x \in A$, xRx.

Example: the less-or-equal to relation on the positive integers

The relation R is anti-reflexive if for every $x \in A$, it is not true that xRx. Example: the less-than relation

The relation R is transitive if for every $x, y, z \in A$, xRy and yRz imply that xRz. Example: the ancestor relation

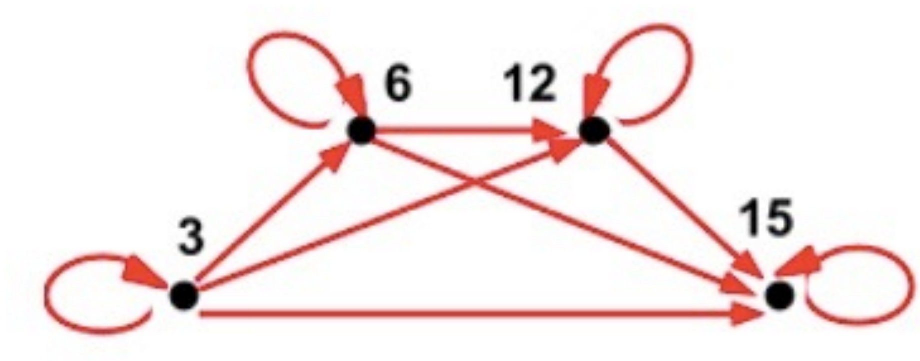
The relation R is symmetric if for every $x, y \in A$, xRy implies that yRx. Example: R = {(a, b) : a,b are actors that have played in the same movie}

The relation R is anti-symmetric if for every $x, y \in A$, xRy and yRx imply that $x = y$. Example: less-or-equal

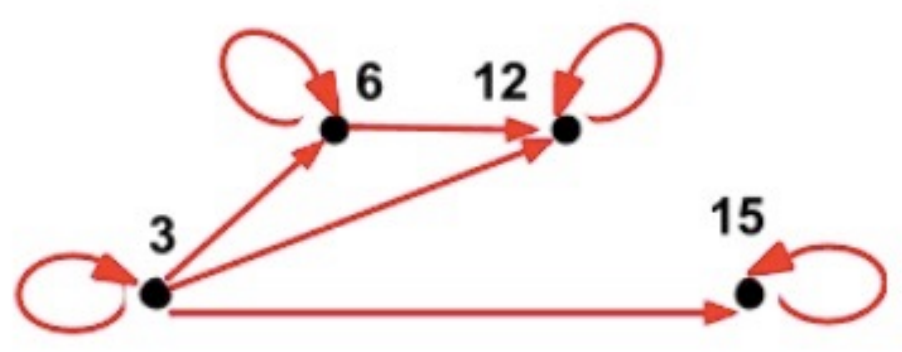
$a \le b$ and $b \le a \rightarrow a = b$

partial orders

Let's look at the graphs for the following relations:



$x \leq y$



x evenly divides y

What properties do these relations have (symmetric, anti-symmetric, reflexive, anti-reflexive, transitive).

partial orders

A relation R on a set *A* is a partial order if it is

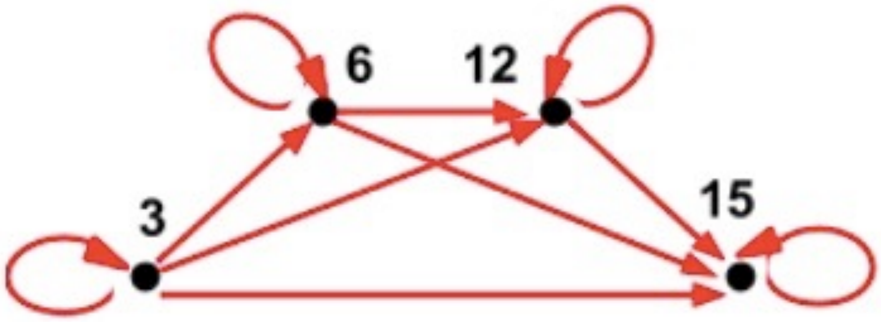
reflexive, transitive, and anti-symmetric.

The notation $a \preceq b$ is used to reflect the fact that a partial order acts like the ≤ operator on the elements of *A*.

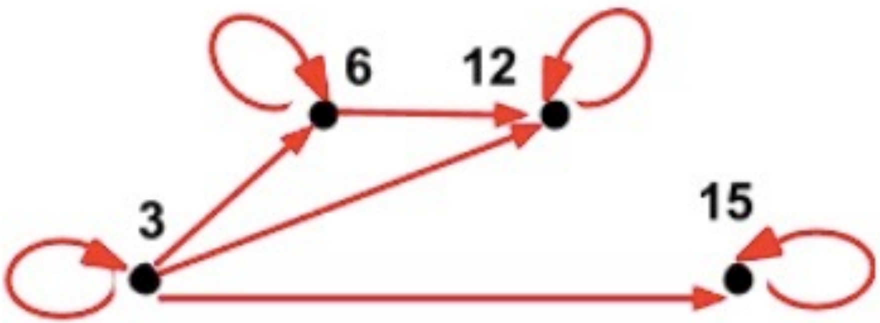
The domain along with a partial order defined on it is denoted $(A, \preceq)$ and is called a partially ordered set or poset.

Example: The ≤ operator acting on the set of integers is a partial order, denoted by (Z, ≤).

partial orders



$x \leq y$



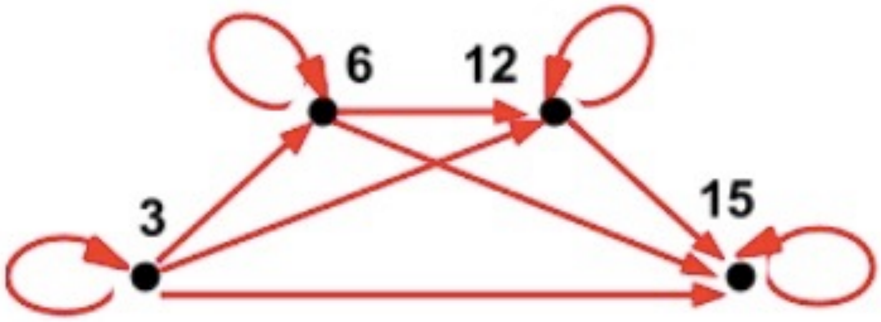
x evenly divides y

Two elements of a partially ordered set, x and y, are said to be comparable if $x \preceq y$ or $y \preceq x$.

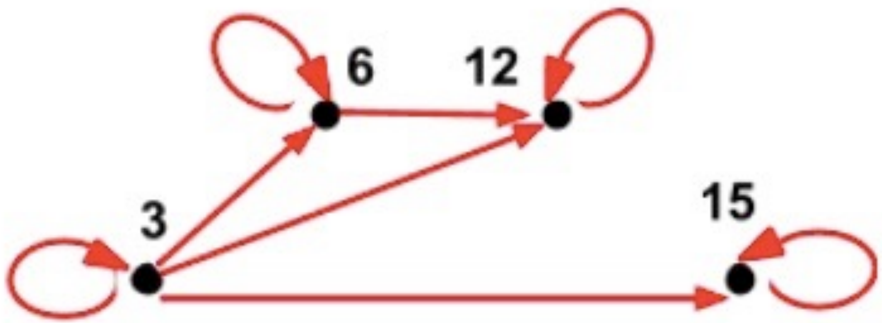
Otherwise they are said to be incomparable.

A partial order is a total order if every two elements in the domain are comparable. The partial order $(Z, \leq)$ is an example of a total order.

partial orders



x ≤ y



x evenly divides y

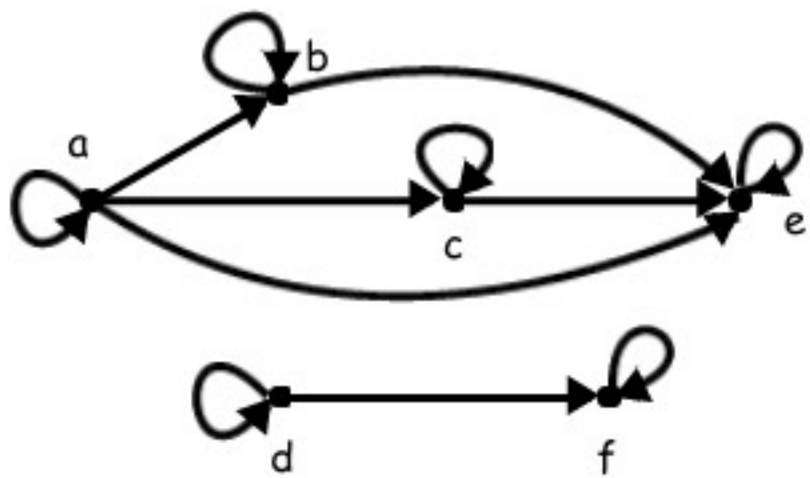
An element x is a minimal element if there is no $y \neq x$ such that $y \preceq x$.

An element x is a maximal element if there is no $y \neq x$ such that $x \preceq y$.

partial orders

Example:

Is the following a partial order?



Properties? reflexive, transitive, anti-symmetric?

What are the minimal/maximal elements?

partial orders

Is the following a partial order?

The domain is a set of students at a school. $x \preceq y$ if x has the same birthday as y.

Is it transitive?

Is it reflexive?

Is it anti-symmetric?

strict orders

A relation R is a strict order if R is transitive and anti-reflexive.

The notation $a \prec b$ is used to express that "a is less than b".

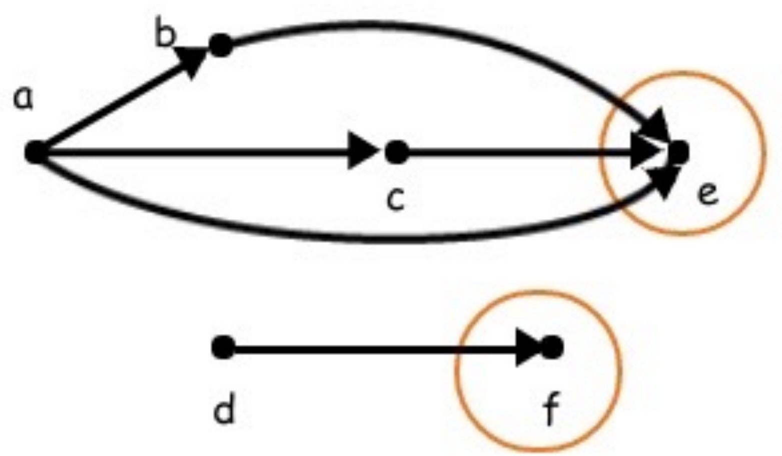
The domain along with the strict order defined on it is called a strictly ordered set and is denoted by $(A, \prec)$.

The definitions of comparable, incomparable, minimal, maximal are the same as for partial orders

strict orders

Example:

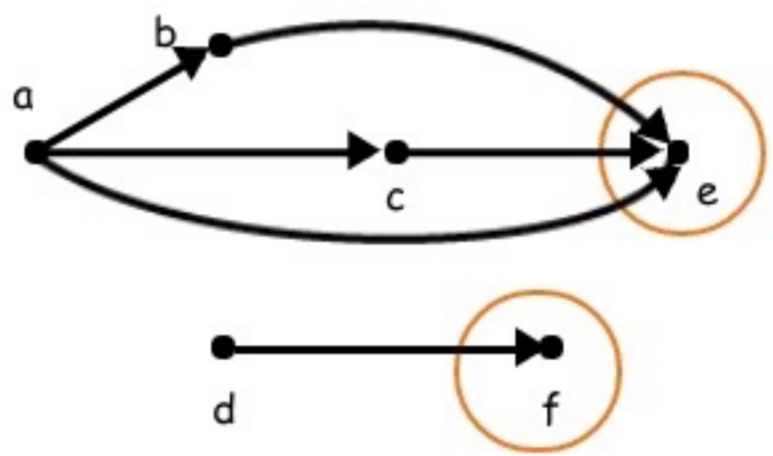
Is the following a strict order?



Properties? anti- reflexive, transitive, anti-symmetric?

strict orders

Example:
Is the following a strict order?



Properties? anti- reflexive, transitive, anti-symmetric?

A relation R that is transitive and anti-reflexive is also anti-symmetric

strict orders

Given a finite set A, let's check if $(P(A), \subset)$ is a strict order.

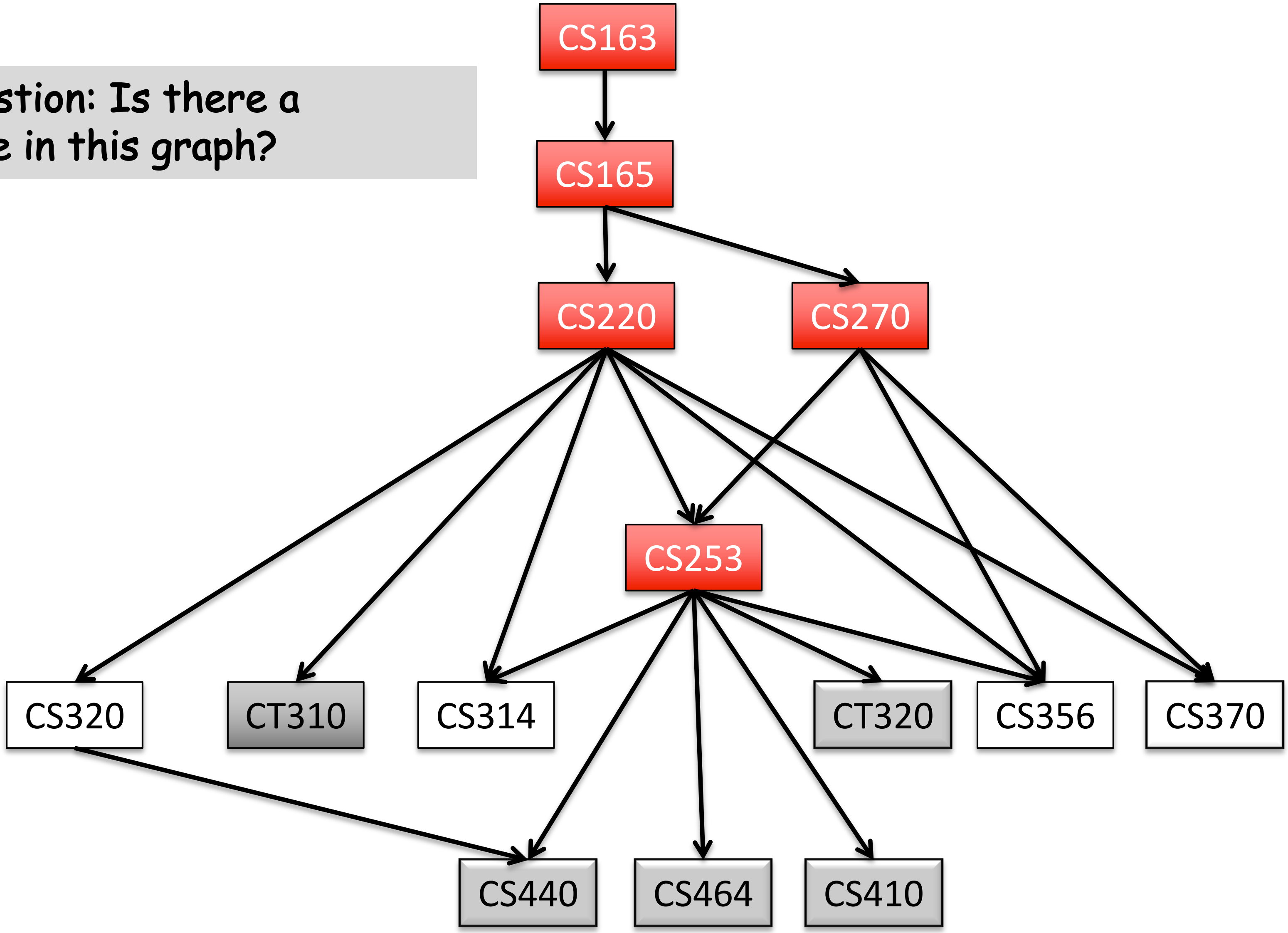
The domain is $P(A)$, the set of all subsets of A.

Two subsets of A, X and Y, are in the relation if $X \subset Y$.

Is it transitive? anti-reflexive?

CS/ACT prerequisite structure

Question: Is there a cycle in this graph?

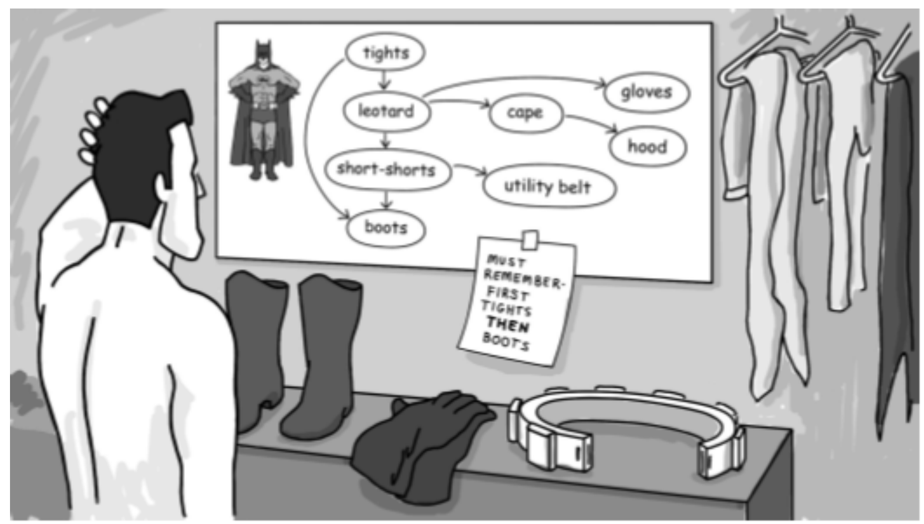


graphs describing precedence

Examples:

- prerequisites for a set of courses
- dependencies between programs (for installation and compilation)

Edge from a to b indicates a should come before b



Batman images are from the book "Introduction to bioinformatics algorithms"

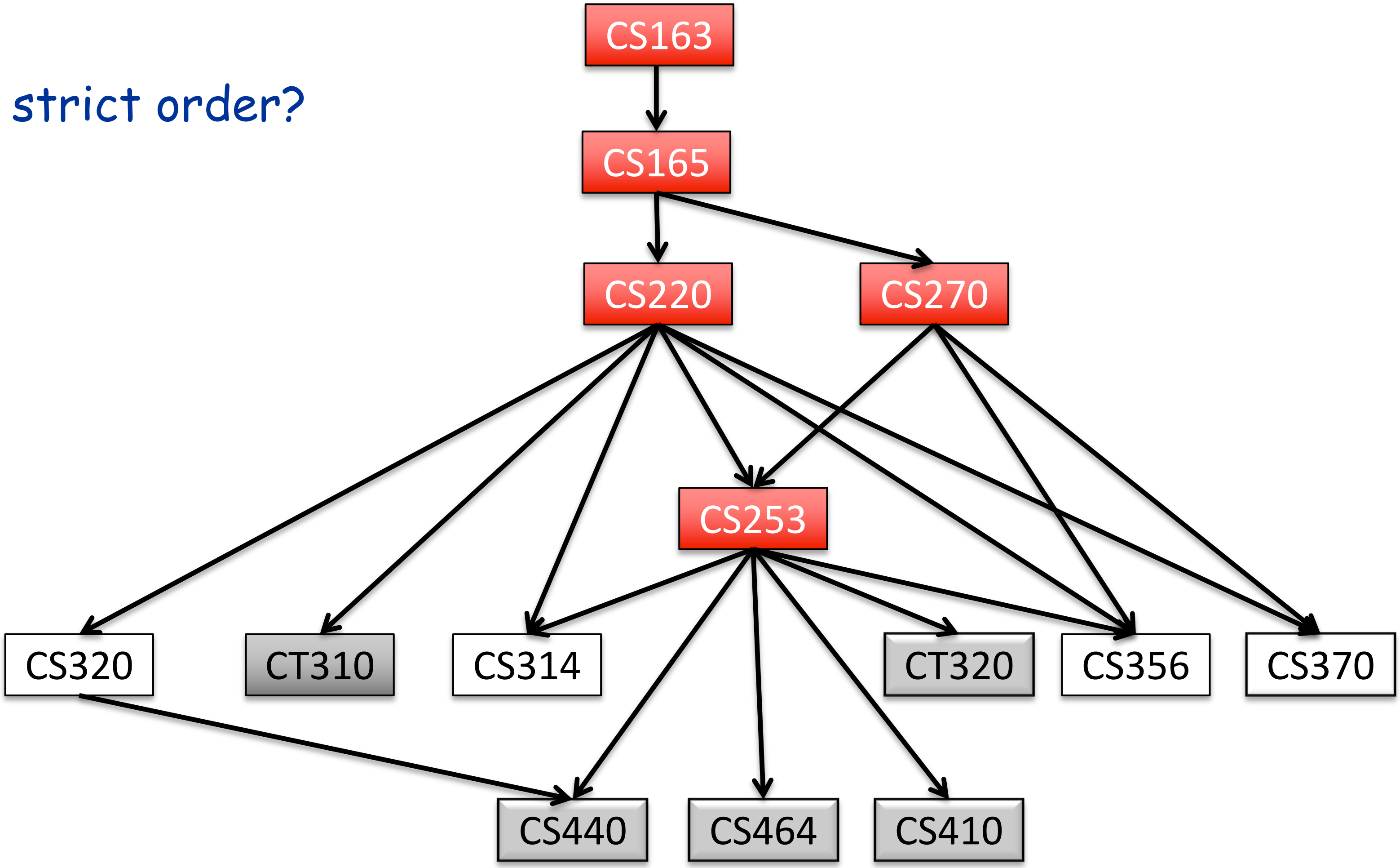
directed acyclic graphs

A directed acyclic graph (DAG) is a directed graph that has no cycles.

directed acyclic graphs

A directed acyclic graph (DAG) is a directed graph that has no cycles.

Is it a strict order?



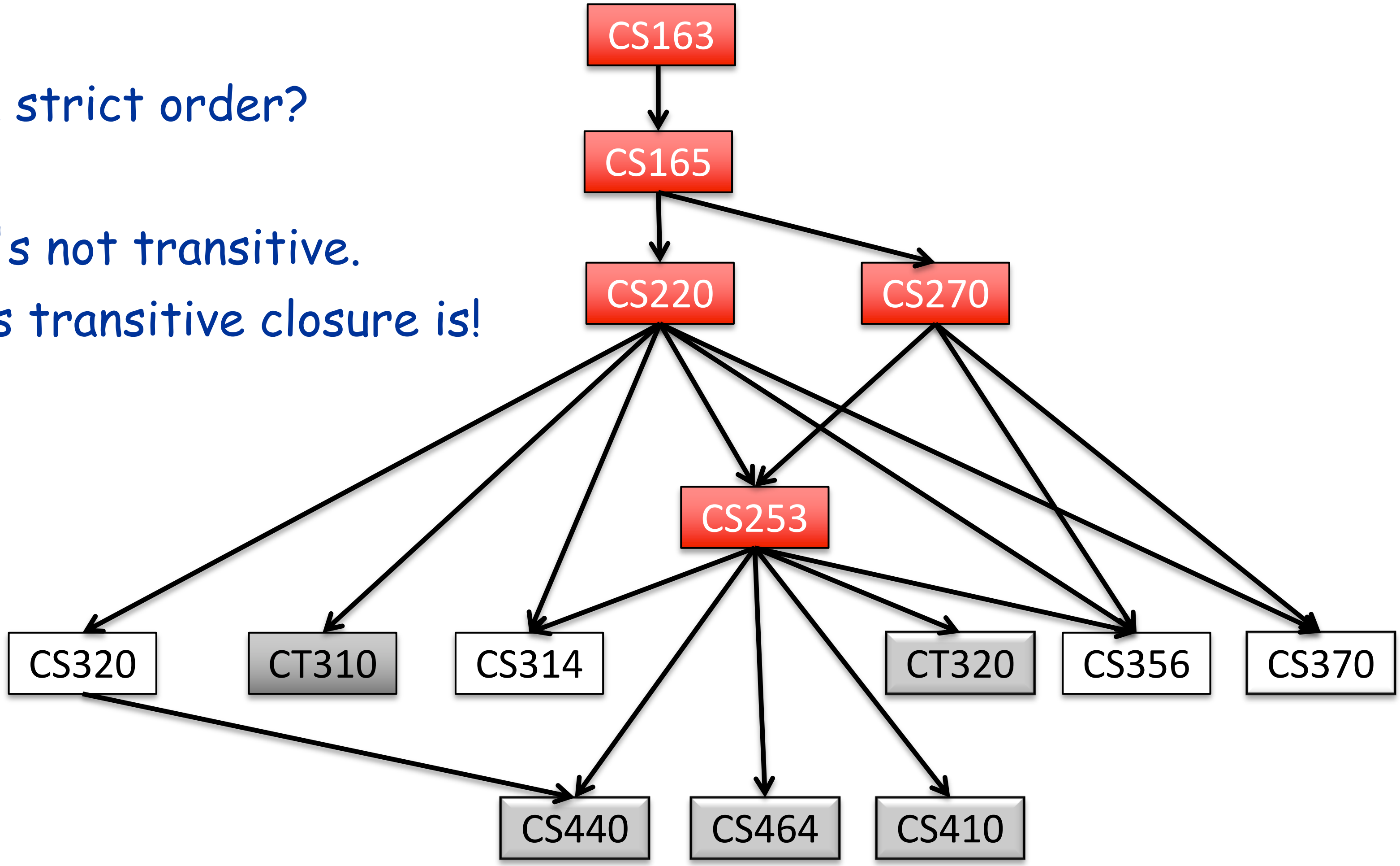
directed acyclic graphs

A directed acyclic graph (DAG) is a directed graph that has no cycles.

Is it a strict order?

No, it's not transitive.

But its transitive closure is!

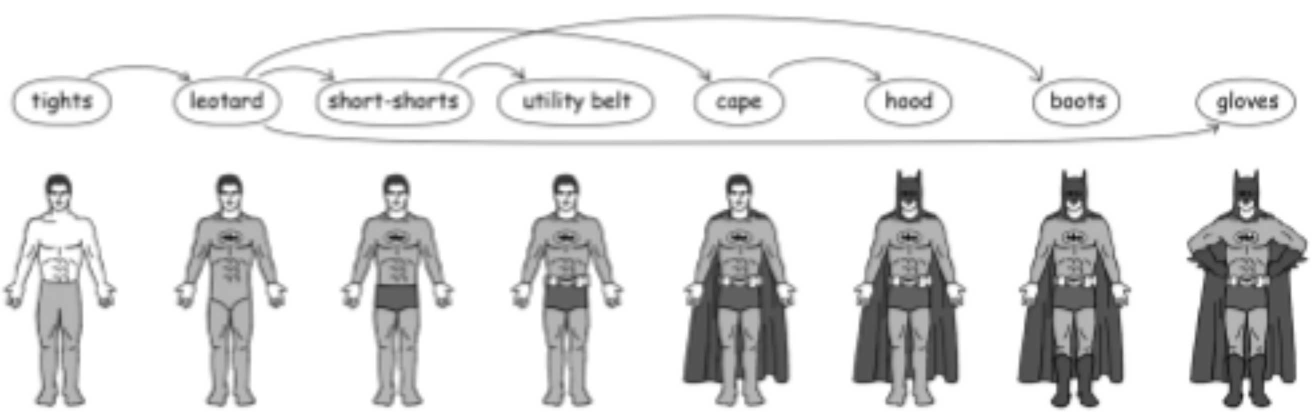


graphs describing precedence

Want an ordering of the vertices of the graph that respects the precedence relation

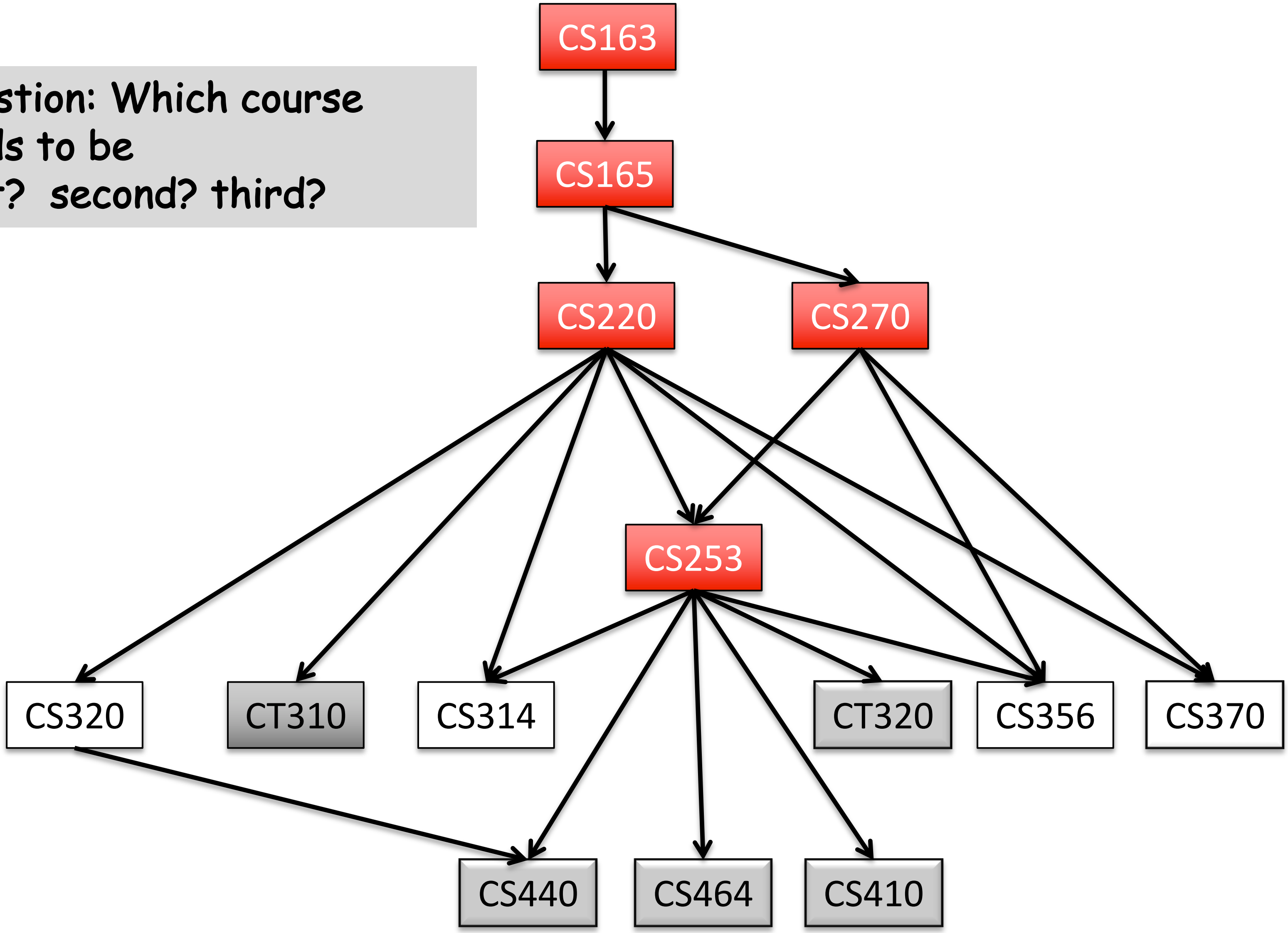
- Example: An ordering of CS courses

Topological sort: listing of nodes such that if (a,b) is an edge, a appears before b in the list



order for CS/ACT courses

Question: Which course needs to be first? second? third?

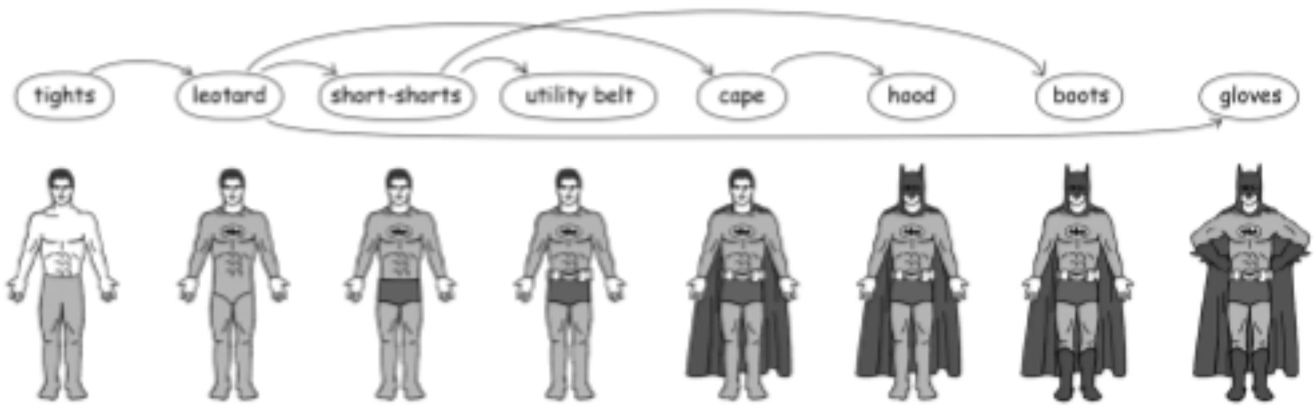


graphs describing precedence

Want an ordering of the vertices of the graph that respects the precedence relation

Topological sort: listing of nodes such that if (a,b) is an edge, a appears before b in the list

Is a topological sort unique?

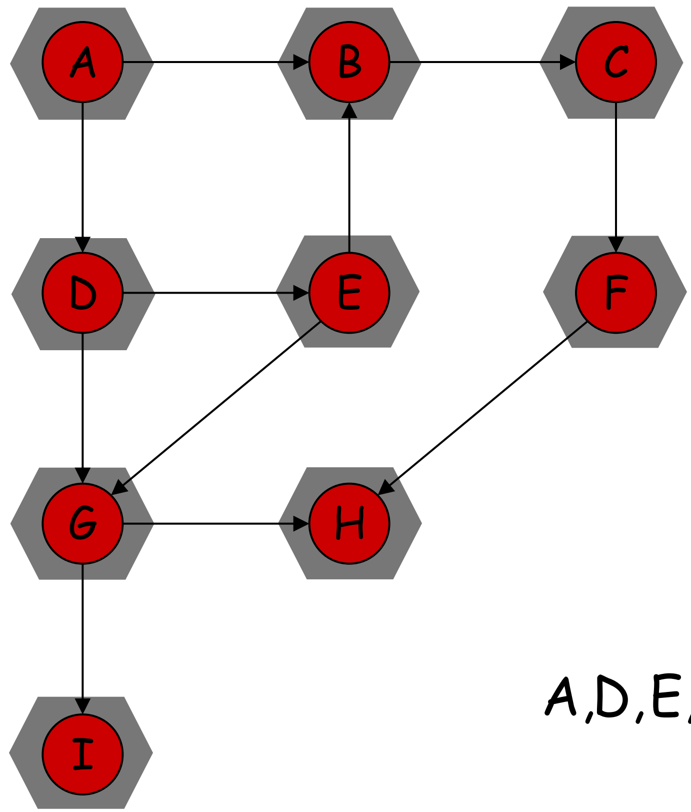


topological sort

- Pick a vertex x with in-degree 0 and remove x from G, including all its outgoing edges.
- Then pick another vertex with in-degree 0 from the remaining vertices.
- Keep selecting vertices until no vertices left.

another algorithm for topological sort

IDEA: nodes with no successors
can be added to the back of the list



A,D,E,B,G,C,F,H,I

directed acyclic graphs

DAGs are an important class of graphs

Used for representing probabilistic relationships between variables (Bayesian networks)

Are at the core of dataflow programming (TensorFlow)

Many computational problems that are NP-hard on general graphs can be solved efficiently on DAGs

n-ary relations

Definition: Let $A_1, A_2, \ldots, A_n$ be sets.

An ***n*-ary relation** on these sets is a subset of

$$A_1 \times A_2 \times \ldots \times A_n.$$

The sets $A_1, A_2, \ldots, A_n$ are called the *domains* of the relation, and *n* is called its *degree*.

Example: The *between* relation consisting of triples (a,b,c) where a,b,c are integers such that $a < b < c$

example

x thinks that y likes z

Person x	Person y	Person z
Alice	Bob	Denise
Charles	Alice	Bob
Charles	Charles	Charles
Denise	Denise	Denise

databases and relations

Students			
StudentName	IDnumber	Major	GPA
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

Databases defined by relations are called *relational databases*