Sphere Intersection

- Sphere centered at $P_c$ with radius $r$.

\[ |P - P_c|^2 = r^2 \]

\[ |L + sU - P_c|^2 - r^2 = 0 \]

substitute

\[ T = P_c - L \]

Yielding a quadratic equation

\[ (sU - T) \cdot (sU - T) - r^2 = 0 \]
Brute Force (II)

• Expand to see what is happening.

\[
\begin{pmatrix}
  s & u_x & t_x \\
  u_y & - & t_y \\
  u_z & - & t_z \\
\end{pmatrix}
\begin{pmatrix}
  s & u_x & t_x \\
  u_y & - & t_y \\
  u_z & - & t_z \\
\end{pmatrix} - r^2 = 0
\]

\[W \cdot W - r^2 = 0\quad W = \begin{pmatrix}
  su_x - t_x \\
  su_y - t_y \\
  su_z - t_z \\
\end{pmatrix} \quad T = \begin{pmatrix}
  x_c - x_0 \\
  y_c - y_0 \\
  z_c - z_0 \\
\end{pmatrix}
\]

\[
(su_x - t_x)^2 + (su_y - t_y)^2 + (su_z - t_z)^2 - r^2 = 0
\]
Sphere Intersection (III)

- Multiply then expand and collect terms.

\[
(u_x^2 + u_y^2 + u_z^2)s^2 +
\]

\[
(-2t_x u_x - 2t_y u_y - 2t_z u_z)s +
\]

\[
(t_x^2 + t_y^2 + t_z^2) - r^2 = 0
\]

- \(U\) is length 1, so \((u_x^2 + u_y^2 + u_z^2) = 1\)
- So the equation may be written as:

\[
s^2 - 2(U \cdot T)s + T \cdot T - r^2 = 0
\]
Reduces to Quadratic

Note vector dot products.

\[ as^2 + bs + c = 0 \]

where

\[ a = 1 \]
\[ b = -2(U \cdot T) \]
\[ c = T^2 - r^2 \]

Therefore:

\[ s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

\[ s = \frac{2(U \cdot T) \pm \sqrt{4(U \cdot T)^2 - 4(T^2 - r^2)}}{2} \]

\[ s = (U \cdot T) \pm \sqrt{(U \cdot T)^2 - T^2 + r^2} \]
Actual Intersection Points

• Compute the two $s$ values for the two intersections:

\[ s_1 = (U \cdot T) + \sqrt{(U \cdot T)^2 - T^2 + r^2} \]
\[ s_2 = (U \cdot T) - \sqrt{(U \cdot T)^2 - T^2 + r^2} \]

8 multiplies/squares (caching results; $r^2$ previously stored)
9 additions/subtractions
1 square root

• Compute the actual positions along the ray for the smallest positive $s$:

\[ s^* = \min(s_1, s_2) \]
\[ P^* = L + s^* U \]

3 multiplies
3 additions
1 min
First Example

Sphere center: $C = (5, 5, 5)$

Ray start: $L = (0, 0, 0)$

Ray direction: $U = \left(\frac{1}{3}\sqrt{3}, \frac{1}{3}\sqrt{3}, \frac{1}{3}\sqrt{3}\right)$

Base to Center: $T = (5, 5, 5)$
Second Example

Sphere center: $C = (5, 5, 5)$

Ray start: $L = (0, 0, 0)$

Ray direction: $U = \left( \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right)$
But Wait

• The proceeding follows naturally from parametric approach to intersection.
• But is it smart? Is there a better way?
• Yes.
  – http://www.cs.unc.edu/~rademach/xroads-RT/RTarticle.html
• Let’s see it,
Faster Method

\[ v^2 + b^2 = c^2 \]
\[ d^2 + b^2 = r^2 \]

\[ d^2 = \left( r^2 - (c^2 - v^2) \right) \]
\[ d = \sqrt{r^2 - (c^2 - v^2)} \]

If \( d^2 \) less than zero, no intersection.
Otherwise, \( Q = E + (v-d)R \)
Faster Method - How Fast?

- Recall how $R$ is defined …
  $$R(s) = L + sU$$
  $$v = (C - L) \cdot U$$

- Need to compute $v$ …
  - 3 multiplies, 2 additions.

- Both $r^2$ and $c^2$ are already computed.
  - $c^2$ computed for case of ray coming from focal point

- Test if $r^2$ is greater than $(c^2 - v^2)$
  - 1 multiply, one conditional.

- Only if intersection, $Q = E + (v - d)R$.
  - 1 subtract, 3 multiplies, 3 additions.

- Another reference on this approach:
  - [http://www.groovyvis.com/other/raytracing/basic.html](http://www.groovyvis.com/other/raytracing/basic.html)
Example 1

Sphere center: $C = (5, 5, 5)$

Ray start: $L = (0, 0, 0)$

Ray direction: $U = \left( \frac{1}{26} \sqrt{26}, \frac{3}{26} \sqrt{26}, \frac{2}{13} \sqrt{26} \right)$

Base to Center: $T = (5, 5, 5)$

$$r^2 - b^2 = -\frac{58}{13}$$
Example 2

Sphere center: \( C = (5, 5, 5) \)

Ray start: \( L = (1, 0, 1) \)

Ray direction: \( U = \left( \frac{1}{6} \sqrt{6}, \frac{1}{3} \sqrt{6}, \frac{1}{6} \sqrt{6} \right) \)

Base to Center: \( T = (4, 5, 4) \)
Option: Rays from Focal Point

• Earlier, we had the ray originate from pixel L.
• With the unit vector pointing from focal point E to pixel L.
• Now, instead, let ray originate from focal point E.

\[ R(s) = L + sU \]

\[ U = \frac{L - E}{\|L - E\|} \]

\[ R(s) = E + sU \]

How might this help?
Intermediate values remain constant across all pixels when always using the focal point E as the base of the ray.
Why Spheres in a Triangle World?

• How to spheres help with big polygonal models??

• Define a sphere around your model.
  – Intersecting the sphere is necessary but not sufficient for intersecting polygons in the model.
  – May help greatly to reduce work.

• What factors contribute to savings?