Lecture 15:
Ray Polygon Intersection

October 18, 2016

Outline

- Universal Basics
  - Polygon edges and tangents
    - Lines, rays, segments, tangents and normals
    - Generating the equation of the plane (infinite)
  - Ray intersection with a plane (infinite)
- Intersection with a convex Polygon
  - Inside-outside tests in 2D
  - Inside-outside tests in 3D
- Odd-even test for a 2D non-convex polygon

Review: A Line

- One dimensional object in n-dimensions
- Ambiguous terminology
  - May be bounded or infinite
  - May be directed or undirected
- Algebraic expression / representation
  - Implicit function
    - Distance from origin and a direction
  - Parametric
    - Two points define a line

Review: A Plane

- Two dimensional object in n-dimensions
- Less ambiguous than Line
  - Generally assumed to be unbounded
- Algebraic expression / representation
  - Implicit function
    - Distance from origin and a direction
  - Parametric
    - Three points define a plane

Review: A Segment

- Like a line, but!
- Term is short for Line Segment.
- Clearly indicates a start and end point.
- Algebraic expression / representation
  - Parametric
  - Generally implicit form is NOT used.
- Consider this question:
  - Do segments S_1 and S_2 intersect?

Review: A Vector

- Direction and magnitude in n-dimensions
- Does NOT have a location in space!
- Where can we draw vectors in illustrations?
  - Anywhere we want, provided it servers our purpose.
- Algebraic expression / representation
  - First choice is an n-tuple of scalars
  - Second choice is a unit length vector and a separate magnitude

Review: A Ray

- One dimensional object in n-dimensions
- Between and infinite line and a segment
  - Bounded at one end only
- Algebraic expression / representation
  - Parameterized line bounded at one end
  - In code, often a tuple (Point, Unit length vector)
  - Is the following a Line, Segment or Ray?

$$P(t) = t(P_2 - P_1) + P_1$$

Review: A Polyline

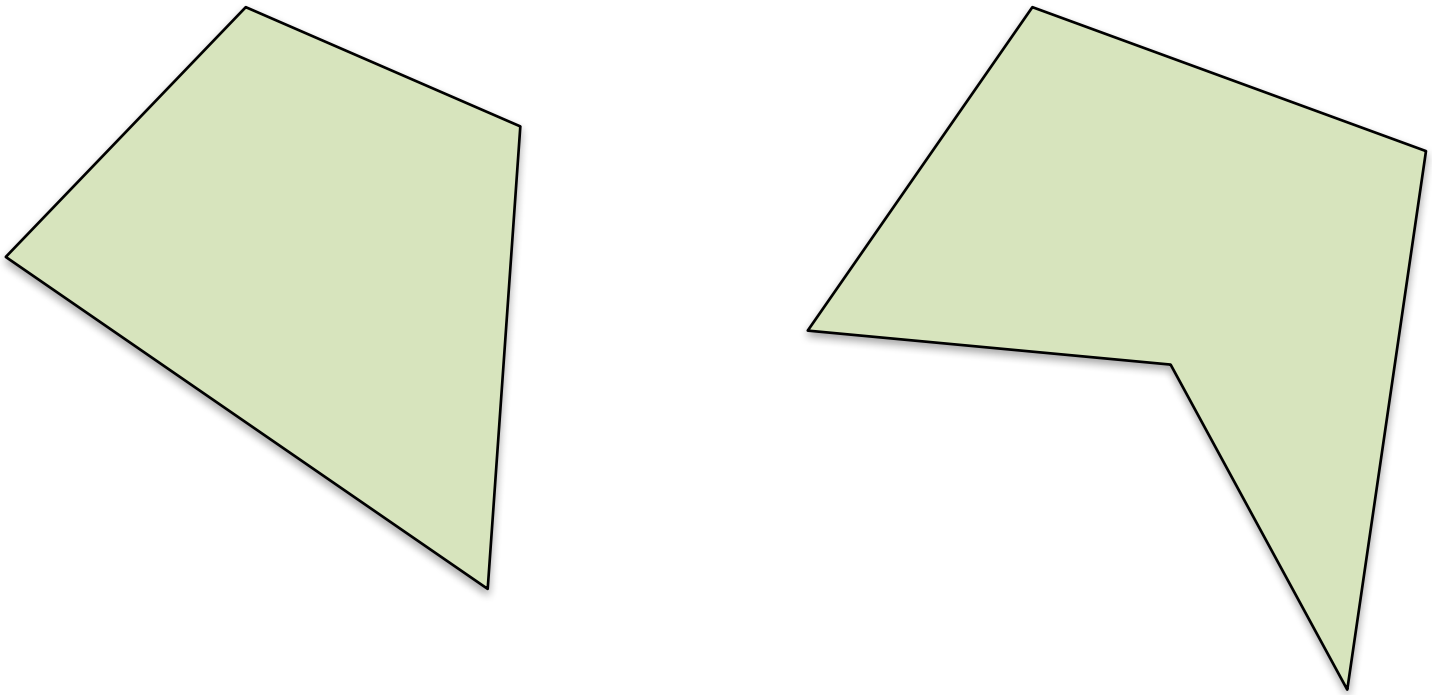
- One dimensional object in n-dimensions
- A sequence of one or more Segments
- Algebraic expression / representation
  - Top-level is a tuple: S1, S2, S3
  - Down one level
    - Segment parameterized between start and end points
  - Shorthand
    - A k Segment Polyline expressed as k+1 Points
- Question
  - What is the area of a Polyline?

Review: A Polygon

- Two dimensional object in n-dimensions
- Algebraic expression / representation
  - Bounded by k Segments
  - Defined by k+1 Points (vertices)
- But testing for inclusion is more involved
  - We cross over into the realm where short algorithms are more helpful than any single algebraic expression.

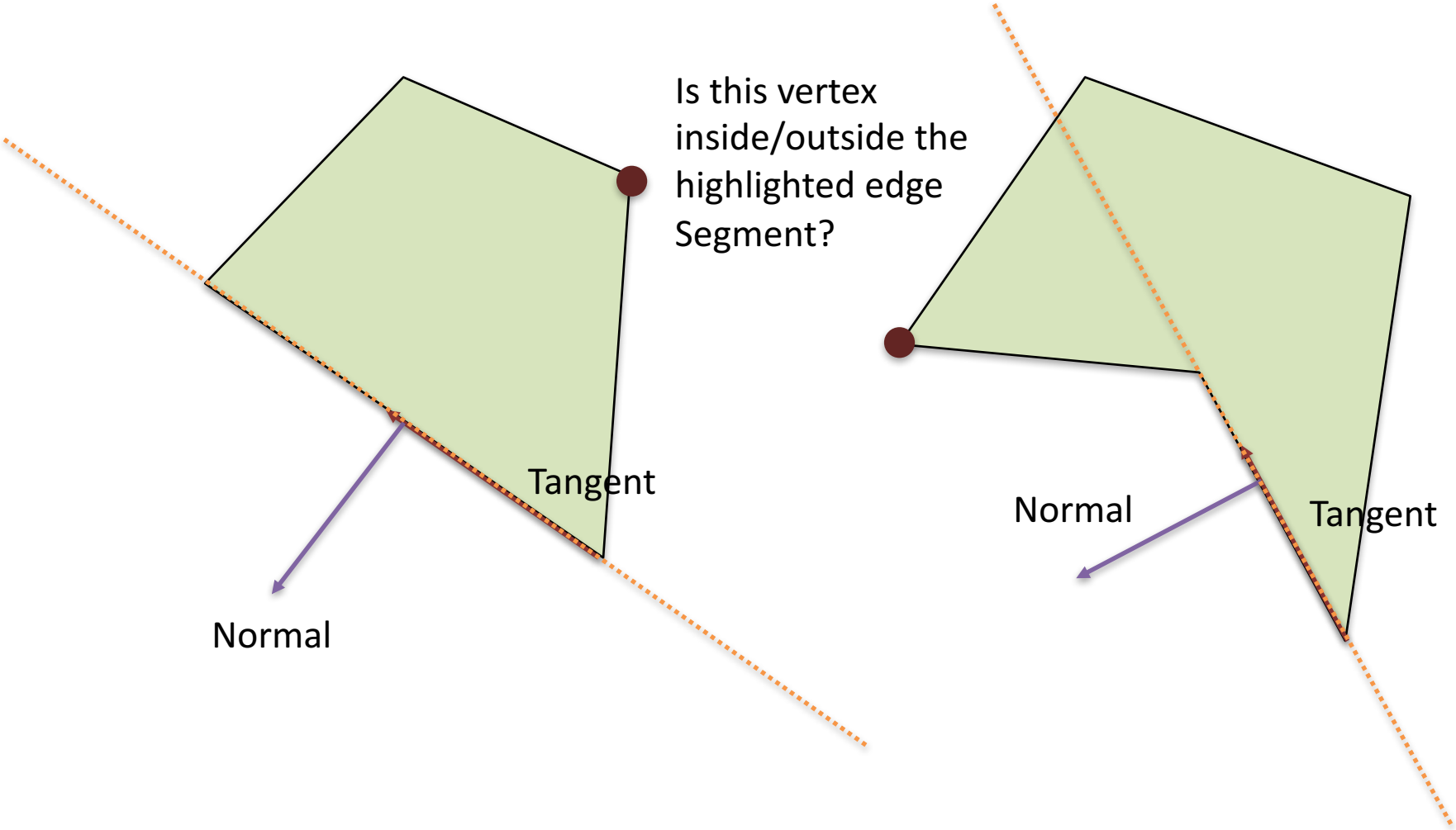
Convexity

- To start, you know it when you see it.



How would you formally confirm (or refute) convexity?

Edge Normal & Edge Tangent



Ray/Polygon Intersection

- All polygons must be planar!
  - How would you test for this condition?
- Intersect rays with…
  - General Polygons
    - Slow, complex
  - Convex Polygons
    - Fast, pretty simple
  - Triangles
    - Faster
    - We've covered this now.

General Ray/Surface Intersection

- Implicit surfaces are defined by

$$f(p) = 0$$

- Given a ray

$$L + tU$$

- The intersection is solved by

$$f\left(L + tU\right) = 0$$

Ray/Polygon Intersection

- For convex polygons there are two parts.
  - Find point of intersection P on infinite plane.
  - Test if point P inside polygon.
- General equation for a plane in 3D:

$$ax + by + cz + d = 0$$

- Therefore an (x,y,z) point P is on a plane iff:

$$N \cdot P = -d$$

- Where

$$N = [a, b, c]$$

Ray/Polygon Intersection (II)

- What is N?
  - It's the surface normal of the plane!
  - All points on a plane project to the same point on the normal vector (so N•P=-d)
- How do we find N?
  - Given vertices {A, B, C…}
  - N = (B-A) × (C-B), then normalize
  - Unless A,B & C are collinear
    - Then N = (C-B) × (D-C)
    - Unless A, B, C and D are collinear…

Ray/Polygon Intersection (III)

- The surface normal N is (a,b,c) in $ax + by + cz + d = 0$
- How do you compute d?
  - Plug in (x,y,z) coordinates of any vertex
- Notes
  - You will need the unit surface normal N of every polygon for ray tracing
  - Therefore, compute N for every polygon when you read it in
  - You only need the d term if you intersect convex polygons (not just triangles)

Planar Intersection

- Given the equation of the plane, calculate where the ray intersects the plane

$$N \cdot \left(L + tU\right) = -d$$

$$tN \cdot U = -\left(N \cdot L + d\right)$$

$$t = \frac{-\left(N \cdot L + d\right)}{N \cdot U}$$

Be careful. What if this is zero?

Ray / Polygon Intersection

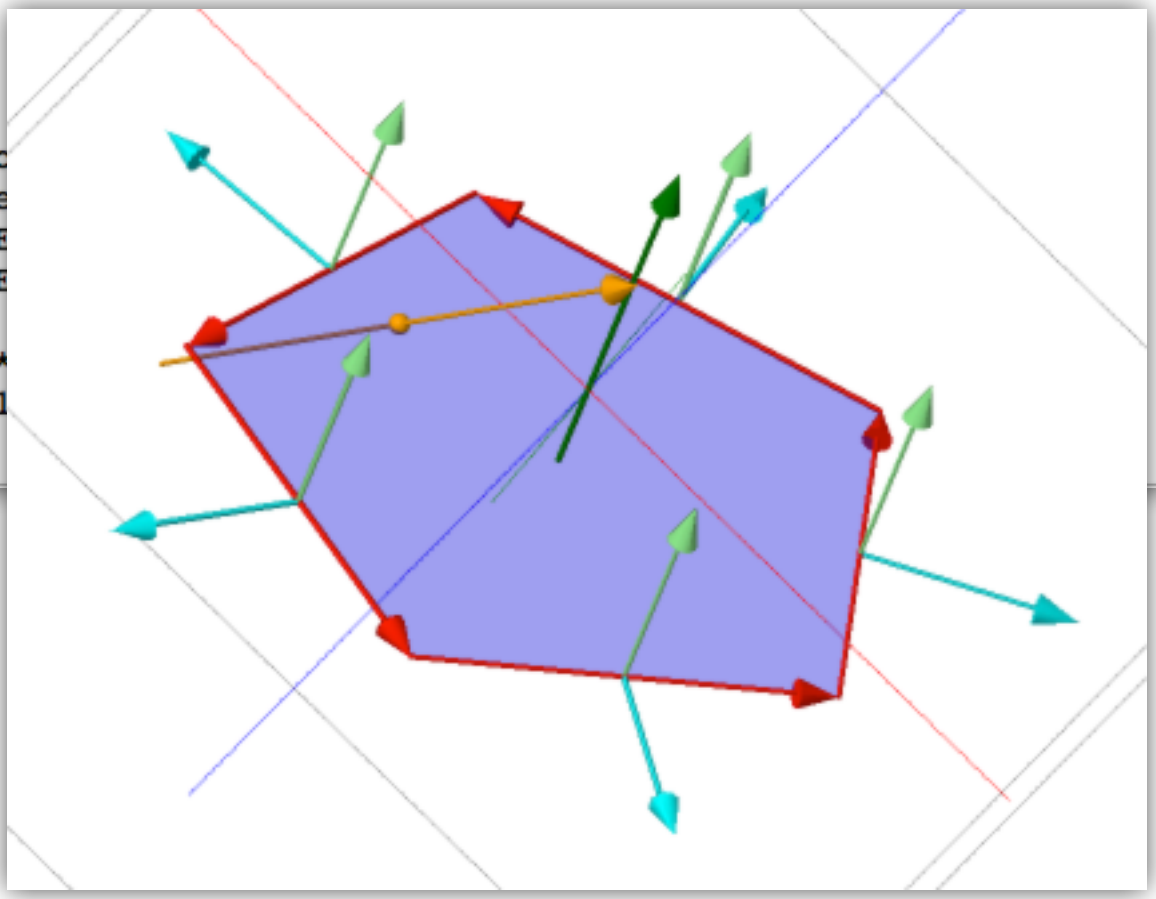
- Given ***t***, we know the point where the ray intersects the plane of the polygon

  $P = L + tU$

- But is it inside the polygon?
- For the convex case see the Sage Notebook.
  - Mostly omitted from this slide deck.
- For non-convex learn the odd-even rule
  - Covered in the remainder of this slide deck.

rayPolygon.sws

```
var('t')
Bv    = vector(SR, (-80,-100,100));  Dv = vector(SR, (1/2,1,-1)); Dv = Dv / Dv.norm()
rho   = SNv.dot_product(VL2[0])
eq1   = SNv.dot_product(Bv + t * Dv) - rho == 0
res   = solve(eq1,t)
tstar = res[0].rhs()
Pint  = Bv + (tstar * Dv)
figcon4  = GPoly3
figcon4 += arrow3d(Pc,(Pc + sc
figcon4 += sum([gedge(e) for e
figcon4 += sum([gnorm(Ed[i], E
figcon4 += sum([gnrmf(Ed[i], E
figcon4 += axes(figb)
figcon4 += arrow3d(Bv,Bv + (2*
figcon4 += point3d(Pint,size=1
figcon4.show()
```



You WILL want to play with this Sage Notebook. There is a great deal going on and you should understand the steps.

Making the 3D problem 2D

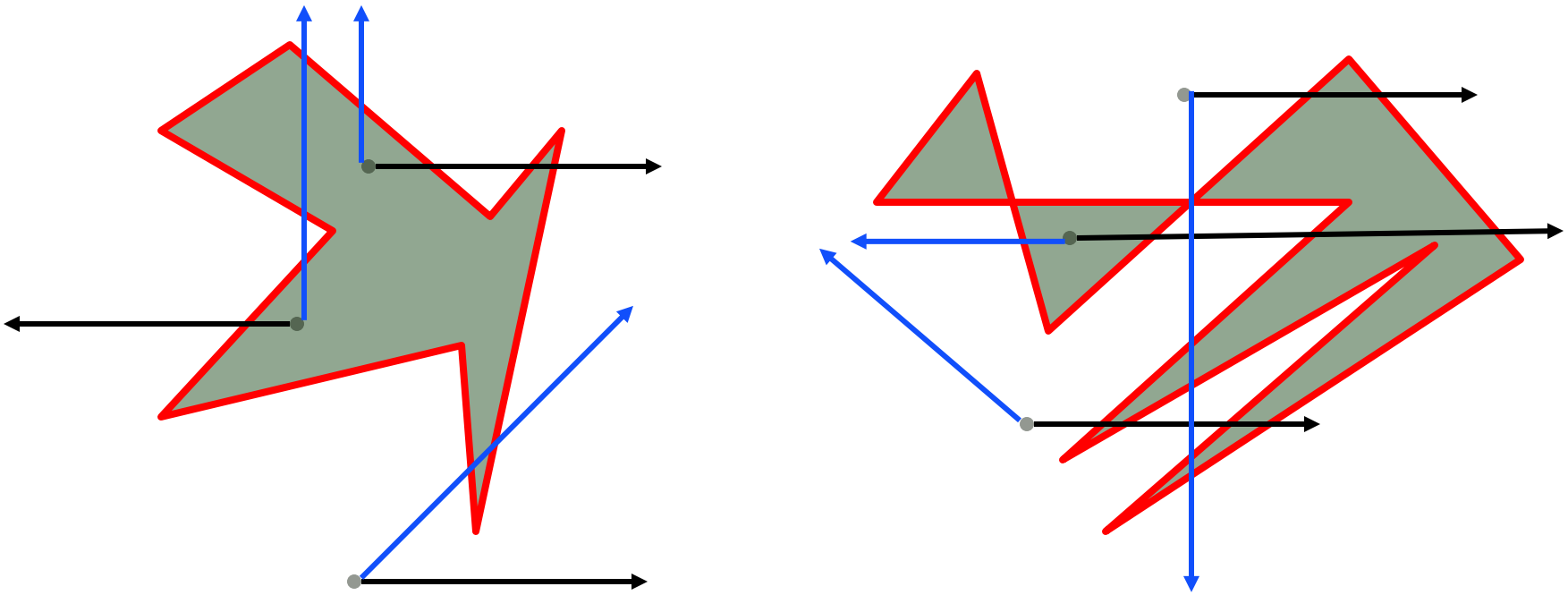
- Once a 3D polygon and 3D point on the plane are found, throw away one dimension
- Note that inside/outside is the same in both cases, provided we don't collapse to a line.
- In these examples, which dimension should be dropped?

$$\left[\begin{pmatrix} \frac{1}{14}\sqrt{14} \\ \frac{1}{7}\sqrt{14} \\ \frac{3}{14}\sqrt{14} \end{pmatrix}, \begin{pmatrix} \frac{4}{29}\sqrt{29} \\ \frac{2}{29}\sqrt{29} \\ \frac{3}{29}\sqrt{29} \end{pmatrix}, \begin{pmatrix} \frac{6}{41}\sqrt{41} \\ \frac{2}{41}\sqrt{41} \\ \frac{1}{41}\sqrt{41} \end{pmatrix}, \begin{pmatrix} \frac{3}{22}\sqrt{22} \\ \frac{3}{22}\sqrt{22} \\ \frac{1}{11}\sqrt{22} \end{pmatrix}\right]$$

Odd/Even Parity Rule

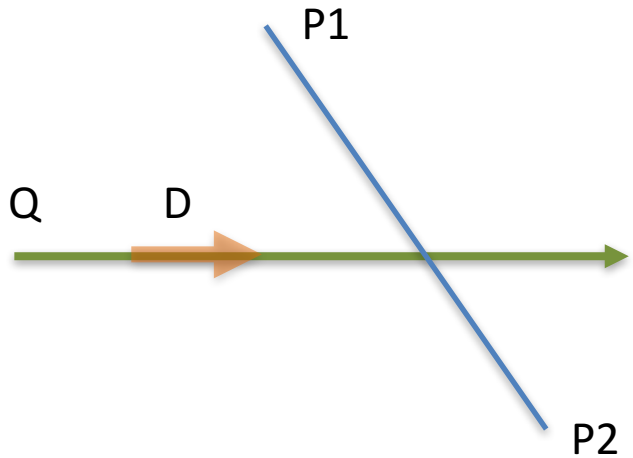
- Tests whether a 2D point P is inside or outside of a polygon
- Step 1: draw a ray from P in any direction in the plane.
- Step 2: count how many boundaries are crossed
  - Odd # of intersections $\Rightarrow$ inside
  - Even # of intersections $\Rightarrow$ outside

Odd/Even Illustrated



Direction doesn't matter!

Full Math – Horizontal Ray




```
var('t','s')
P1 = matrix(SR, 2,1, var('px1','py1'))
P2 = matrix(SR, 2,1, var('px2','py2'))
Q  = matrix(SR, 2,1, var('qx','qy'))
D  = matrix(SR, 2,1, (1,0))
P = (P2 - P1) * t + P1
R = Q + s * D
eq1 = P[0,0] == R[0,0]
eq2 = P[1,0] == R[1,0]
res = solve([eq1,eq2],s,t)
pretty_print(eq1)
pretty_print(eq2)
pretty_print(res[0][1], res[0][0])
```

$$-(px_1 - px_2)t + px_1 = qx + s$$

$$-(py_1 - py_2)t + py_1 = qy$$

$$t = \frac{py_1 - qy}{py_1 - py_2} s = \frac{px_2(py_1 - qy) - px_1(py_2 - qy) - (py_1 - py_2)qx}{py_1 - py_2}$$

Interesting Consequence

- Consider the following Polygon filled by PowerPoint: