Lecture 14:
Ray Sphere Intersection

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Sphere Intersection

- Sphere centered at $P_c$ with radius $r$.

\[ |P - P_c|^2 = r^2 \]

\[ |L + sU - P_c|^2 - r^2 = 0 \]

Substitute

\[ T = P_c - L \]

Yielding a quadratic equation

\[ (sU - T) \cdot (sU - T) - r^2 = 0 \]
**Brute Force (II)**

- Expand to see what is happening.

\[
\begin{pmatrix}
  u_x & t_x \\
  s & u_y - t_y \\
  u_z & t_z
\end{pmatrix}
\begin{pmatrix}
  u_x & t_x \\
  s & u_y - t_y \\
  u_z & t_z
\end{pmatrix} - r^2 = 0
\]

\[
W \cdot W - r^2 = 0 \\
W = \begin{pmatrix}
  su_x - t_x \\
  su_y - t_y \\
  su_z - t_z
\end{pmatrix} \\
T = \begin{pmatrix}
  x_c - x_0 \\
  y_c - y_0 \\
  z_c - z_0
\end{pmatrix}
\]

\[
(su_x - t_x)^2 + (su_y - t_y)^2 + (su_z - t_z)^2 - r^2 = 0
\]
Sphere Intersection (III)

- Multiply then expand and collect terms.

\[
\begin{align*}
(u_x^2 + u_y^2 + u_z^2)s^2 + \\
(-2t_xu_x - 2t_yu_y - 2t_zu_z)s + \\
(t_x^2 + t_y^2 + t_z^2) - r^2 &= 0
\end{align*}
\]

- U is length 1, so \((u_x^2 + u_y^2 + u_z^2) = 1\)

- So the equation may be written as:

\[
s^2 - 2(U \cdot T)s + T \cdot T - r^2 = 0
\]
Reduces to Quadratic

Note vector dot products.

\[ as^2 + bs + c = 0 \]

where

\[ a = 1 \]
\[ b = -2(U \cdot T) \]
\[ c = T^2 - r^2 \]

Therefore:

\[ s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

\[ s = \frac{2(U \cdot T) \pm \sqrt{4(U \cdot T)^2 - 4(T^2 - r^2)}}{2} \]

\[ s = (U \cdot T) \pm \sqrt{(U \cdot T)^2 - T^2 + r^2} \]
Actual Intersection Points

- Compute the two $s$ values for the two intersections:

$$s_1 = (U \cdot T) + \sqrt{(U \cdot T)^2 - T^2 + r^2}$$

$$s_2 = (U \cdot T) - \sqrt{(U \cdot T)^2 - T^2 + r^2}$$

- Compute the actual positions along the ray for the smallest positive $s$:

$$s^* = \min(s_1, s_2)$$

$$P^* = L + s^*U$$

- 8 multiplies/squares (caching results; $r^2$ previously stored)
- 9 additions/subtractions
- 1 square root

- 3 multiplies
- 3 additions
- 1 min
First Example

Sphere center: \[ C = (5, 5, 5) \]

Ray start: \[ L = (0, 0, 0) \]

Ray direction: \[ U = \left( \frac{1}{3} \sqrt{3}, \frac{1}{3} \sqrt{3}, \frac{1}{3} \sqrt{3} \right) \]

Base to Center: \[ T = (5, 5, 5) \]
Second Example

Sphere center: \( C = (5, 5, 5) \)

Ray start: \( L = (0, 0, 0) \)

Ray direction: \( U = \left( \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right) \)
But Wait

• The proceeding follows naturally from parametric approach to intersection.
• But is it smart? Is there a better way?
• Yes.
  – http://www.cs.unc.edu/~rademach/xroads-RT/RTarticle.html
• Let’s see it,
Faster Method

\[ v^2 + b^2 = c^2 \]
\[ d^2 + b^2 = r^2 \]
\[ d^2 = \left( r^2 - (c^2 - v^2) \right) \]
\[ d = \sqrt{r^2 - (c^2 - v^2)} \]

If \( d^2 \) less than zero, no intersection. Otherwise, \( Q = E + (v-d)R \)
Faster Method - How Fast?

• Recall how $R$ is defined …  
  \[ R(s) = L + sU \]

• Need to compute $v$ …
  – 3 multiplies, 2 additions.

• Both $r^2$ and $c^2$ are already computed.
  – $c^2$ computed for case of ray coming from focal point

• Test if $r^2$ is greater than $(c^2 - v^2)$
  – 1 multiply, one conditional.

• Only if intersection, \( Q = E + (v-d)R \).
  – 1 subtract, 3 multiplies, 3 additions.

• Another reference on this approach:
  – [http://www.groovyvis.com/other/raytracing/basic.html](http://www.groovyvis.com/other/raytracing/basic.html)
Example 1

Sphere center: \( C = (5, 5, 5) \)

Ray start: \( L = (0, 0, 0) \)

Ray direction: \( U = \left( \frac{1}{26} \sqrt{26}, \frac{3}{26} \sqrt{26}, \frac{2}{13} \sqrt{26} \right) \)

Base to Center: \( T = (5, 5, 5) \)

\[
r^2 - b^2 = -\frac{58}{13}
\]
Example 2

Sphere center: \( C = (5, 5, 5) \)
Ray start: \( L = (1, 0, 1) \)
Ray direction: \( U = \left( \frac{1}{6} \sqrt{6}, \frac{1}{3} \sqrt{6}, \frac{1}{6} \sqrt{6} \right) \)
Base to Center: \( T = (4, 5, 4) \)
Option: Rays from Focal Point

• Earlier, we had the ray originate from pixel L.

• With the unit vector pointing from focal point E to pixel L.

• Now, instead, let ray originate from focal point E.

\[ R(s) = L + sU \]

\[ U = \frac{L - E}{\|L - E\|} \]

\[ R(s) = E + sU \]

How might this help?
Intermediate values remain constant across all pixels when always using the focal point E as the base of the ray.
Why Spheres in a Triangle World?

• How do spheres help with big polygonal models??

• Define a sphere around your model.
  – Intersecting the sphere is necessary but not sufficient for intersecting polygons in the model.
  – May help greatly to reduce work.

• What factors contribute to savings?