Know what radiosity computation does. Do not expect to implement nor see underlying equations this semester.
Now – The Journey to 2D Rotation

\[
\begin{bmatrix}
\cos 90^\circ & \sin 90^\circ \\
-sin 90^\circ & \cos 90^\circ
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]
Vectors, Points & Matrices

• The geometry for graphics rests upon
  – Scalars, Vectors, Points, and Matrices
• And why? The short answer.
  – Objects are collections of points
  – Light rays are vectors
  – Objects & Light interact in Euclidean spaces
  – Placement in space is done using matrices
• Now for the longer answer…
But let’s start with… Scalars

• Scalar - a number.
  – Two Operations -
    • Addition, Multiplication.
  – Axioms
    • Associative
    • Commutative
    • Invertible
  – Invertible implies
    • Subtraction
    • Division

\[ \alpha + \beta = \beta + \alpha \]
\[ \alpha \cdot \beta = \beta \cdot \alpha \]
\[ \alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma \]
\[ \alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma \]
\[ \alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma \]
Vectors

- Vector - direction and magnitude
  - Two Operations -
    - Scalar-vector multiplication
    - Vector-vector addition
  - Often expressed as an n-tuple of scalars.

\[ \mathbf{v} = [v_1, v_2, v_3, \ldots, v_n] \]
Test: Do you Get It?

• Are these two vectors the same?
Vector Spaces

• Combinations of vectors generate new vectors.

\[ u = \alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \alpha_3 \cdot v_3 \]

for example ...

\[
\begin{bmatrix}
3 \\
4 \\
1
\end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + 4 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
\]
or

\[
\begin{bmatrix}
3 \\
0 \\
1
\end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 4 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]
Key Vector Space Concepts

• **Span**
  – The space of all vectors that can be created by linear combinations of a set of vectors

• **Basis Vectors**
  – A set of vectors that span a space
  – Generally focus on basis vectors that are
    • Orthogonal to each other (independent axes)
    • Unit length

• **Orthonormal**
  – Fancy term for a set of vectors that are mutually orthogonal and unit length
Orthogonal?
Vectors beg “Where are we?”  

*Directly over the center of the Earth?*

- More seriously, vector spaces lack location
  - Location requires an origin: a reference.
- Vector spaces have no origin.
- Now let us introduce points.
  - A point is not the same thing as a vector!
- New operations
  - Point-point subtraction yields a vector.
  - A point plus a vector yields a point.
Point + Vector = Point

– Linear combinations of basis vectors
– … and a specified origin - a point.

\[ P = O + \alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \alpha_3 \cdot v_3 \]

for example ...

\[
\begin{pmatrix}
7 \\
4 \\
3
\end{pmatrix} = \begin{pmatrix}
2 \\
2 \\
2
\end{pmatrix} + 5 \cdot \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix} + 2 \cdot \begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix} + 1 \cdot \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}
\]

Typically, we think of the origin as being at [0,0,0], but that somewhat confuses the real meaning of an origin.

With an origin, you always know where you are (relatively).
But Wait, …

• Do Points Exist Without Coordinates?
  • The answer is – yes!
    – We are adopting the physicists view.
• Why does this matter …

In graphics, keeping the intrinsic geometry of objects separate from their coordinate manifestation in a particular frame of reference is essential.
Same Point - By Example

- The Sphere has intrinsic properties.
  - Independent of reference frame A (or B).

Reference Frame A

Reference Frame B
Intrinsic Properties

• What matters is the relation of the data to the reference frame.
  – Moving the sphere toward the reference point is the same as moving the reference point toward the sphere
  – The same sphere can be expressed in different reference frames

• “World Coordinates” aren’t special
  – As long as all the data is expressed relative to the same reference frame
Where is a Point Revisited

• To specify a point in a Euclidean space.

\[ P = O + x v_1 + y v_2 \]

\[
\begin{vmatrix}
    p_x \\
    p_y \\
\end{vmatrix} = \begin{vmatrix}
    x_o \\
    y_o \\
\end{vmatrix} + \begin{vmatrix}
    1 \\
    0 \\
\end{vmatrix} + \begin{vmatrix}
    0 \\
    1 \\
\end{vmatrix}
\]
A Point named Fred

Which is it? $$\text{Fred} = \begin{vmatrix} 6 \\ 6 \end{vmatrix} \text{ or } \text{Fred} = \begin{vmatrix} 4 \\ 3 \end{vmatrix}$$
2D Translation

• Think about the previous example
• Can you decide between
  – Fred was moved down and to the left.
  – Reference frame was moved up and to the right.
• Generally you cannot
• More important

Often in graphics it is equally valid, or even preferable, to think of movement as shifting a reference frame rather than moving an object.
2D Translation - Moving Fred

\[ x = 6 \quad \text{and} \quad u = 4 \quad \text{as translation} \quad \begin{vmatrix} 6 \\ 6 \end{vmatrix} = \begin{vmatrix} 4 \\ 3 \end{vmatrix} + \begin{vmatrix} 2 \\ 3 \end{vmatrix} \]
Affine to Euclidean Space

• Euclidean Space adds to affine space a new operation, the *dot product*.

• You all know the algebraic definition.

\[ u \cdot v = \sum_{i} u_i v_i \quad |v| = \sqrt{v \cdot v} \]

• Do you know its geometric interpretation?

\[ u \cdot v = |u||v|\cos(\theta) \]

• If the dot product is zero, the vectors are orthogonal.
Dot Product Yields Projection

• To start, set origin at zero. Now observe.

\[
\begin{bmatrix}
  p_x \\
  p_y
\end{bmatrix}
= x \begin{bmatrix}
  1 \\
  0
\end{bmatrix} + y \begin{bmatrix}
  0 \\
  1
\end{bmatrix} \implies x = \begin{bmatrix}
  p_x \\
  p_y
\end{bmatrix} \cdot \begin{bmatrix}
  1 \\
  0
\end{bmatrix}, \quad y = \begin{bmatrix}
  p_x \\
  p_y
\end{bmatrix} \cdot \begin{bmatrix}
  0 \\
  1
\end{bmatrix}
\]

The distance of a point from the origin along a dimension, i.e. along a basis vector, is measured by a dot product between the point and the basis vector.
Know & Love Dot Products 1

- An easy way to define a line ...

\[ L \Rightarrow F(x, y) = 0 \]
\[ n \cdot L - \rho = 0 \]
\[ n \cdot n = 1 \]
\[ \rho = n \cdot P = n_x p_x + n_y p_y \]
\[ \begin{vmatrix} n_x & x \end{vmatrix} - \rho = 0 \]
\[ \begin{vmatrix} n_y & y \end{vmatrix} \]
\[ n_x x + n_y y - \rho = 0 \]
And in 3D

Riddle: What do you call all points a distance of 3 from the origin measured in a direction defined by a vector $n$?

\[
P = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad n = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}
\]

\[
F = \frac{2}{3} x + \frac{2}{3} y + \frac{1}{3} z - 3
\]
Above you see how almost all texts and courses introduce 2D rotation. This is entirely correct, but there is a more intuitive way to understand rotation.
Know & Love Dot Products 2

- Consider an alternate basis

\[ |x| = u \begin{vmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{vmatrix} + v \begin{vmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{vmatrix} \]

\[ |v| = \begin{vmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{vmatrix} |y| \]
Welcome to 2D Rotation

\[
\begin{vmatrix}
|u| = \begin{vmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{vmatrix}
\begin{vmatrix}
x \\
y
\end{vmatrix}
\]

\[
|u| = \begin{vmatrix}
\cos(45^\circ) & \sin(45^\circ) \\
-\sin(45^\circ) & \cos(45^\circ)
\end{vmatrix}
\begin{vmatrix}
x \\
y
\end{vmatrix}
\]

These are the same!
Rotate by $\theta$

$$M = RP$$

$$R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \quad P = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} \cos(\theta) x - \sin(\theta) y \\ \sin(\theta) x + \cos(\theta) y \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Does this make sense, given the geometry of the dot product?
More Standard Approach

Derivation of Rotation Matrix

\[ x_1 = r \cos(\theta) \]
\[ y_1 = r \sin(\theta) \]
\[ x_2 = r \cos(\theta + \phi) \]
\[ y_2 = r \sin(\theta + \phi) \]
Derivation (cont.)

**Magic Trig Identity:**

\[
\begin{align*}
\cos(a + b) &= \cos(a)\cos(b) - \sin(a)\sin(b) \\
\sin(a + b) &= \sin(a)\cos(b) + \sin(b)\cos(a)
\end{align*}
\]

\[
x_2 = r \cos(\theta + \phi)
\]

\[
x_2 = r \cos(\theta)\cos(\phi) - r \sin(\theta)\sin(\phi)
\]

\[
x_2 = x_1 \cos(\phi) - y_1 \sin(\phi)
\]

*Note: the process for \(y_2\) is the same*
The End