Lecture 03:
2D Transformations

August 28, 2018
Previous Lecture & Today

• Last Thursday
  – Scalars, Vectors and Points
  – Vector Spaces
  – Affine Spaces
  – Euclidean Spaces
  – Know and love the dot product

• Today
  – Projection (dot product) and rotation,
  – Homogeneous Coordinates for 2D

• … and SageMath!
About SageMath

SageMath is a free open-source mathematics software system licensed under the GPL. It builds on top of many existing open-source packages: NumPy, SciPy, matplotlib, SymPy, Maxima, GAP, FLINT, R and many more. Access their combined power through a common, Python-based language or directly via interfaces or wrappers.

Mission: Creating a viable free open source alternative to Magma, Maple, Mathematica and Matlab.

Do you want to learn how to use SageMath?
Read Sage for Undergraduates by Gregory Bard or Mathematical Computation with Sage by Paul Zimmermann et al. translations: Calcul mathématique avec Sage (French), Rechnen mit Sage (German)

CoCalc (SageMathCloud)
or: SageMathCell

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Getting Notebooks

For the most part this page is used to distribute SageMath notebooks used to illustrate key concepts. Students must during the course of the semester download and actually run these notebooks. It is not enough to passively review them, it is essential to get into the examples, make changes, and learn from interaction.

Note in particular there are links to the 'dead' HTML output of a notebook. This is useful for a quick read through of a notebook. However, to actually run the notebook download the IPYNB file and upload it to a SageMath Jupyter Server. Server software is installed on the CS Department Machines and you may also download it to your own machine.

Lecture 3 Notebook 1
Notebook cs410lec03n01 illustrates the dot product as a projection operator in 2D. In other words, measuring the distance from the origin to a point in the direction of a vector u.

Zipped IPYNB Notebook File
Dot Product as Projection

The linear algebraic definition of a dot product is clearly taught everywhere. Namely, the sum of pairwise products between first, second, third values etc. However, one of the more important underlying geometric meanings of the dot product is sometimes neglected. Not here. Much of what we do in computer graphics depends critically upon a reflexive understanding of the dot product as a projection operator. To better understand what we mean by 'projection', read on.

Ross Beveridge
August 28, 2018

As will be common in these notebooks, the next sequence of commands configure options for running the notebook such as how to display math, etc.

```python
In [1]:
#display latex
latex.matrix_delimiters(left='|', right='|')
latex.vector_delimiters(left='[', right=']')
```

To begin, here are the basics starting with two vectors.

```python
In [2]:
var('a', 'b', 'c', 'd')
u = vector([a, b])
v = vector([c, d])
pretty_print("u = ", u, ", v = ", v)
pretty_print(LatexExpr("u \cdot v = ") + v.dot_product(u))
```

\[ u = [a, b], \ v = [c, d] \]
\[ u \cdot v = a \cdot c + b \cdot d \]

If the vector \( u \) is of unit length, and the vector \( v \) represents the position of a point \( Q \) in space, then the dot product of \( u \) and \( v \)...
SageMath Rotation is Projection

```python
In [5]:
    bnd = 5.0
    gu = arrow((0,0),u)
    gv = arrow((0,0),v)
    gud = line(((0,0),bnd*u), linestyle="--")
    gvd = line(((0,0),bnd*v), linestyle="--")
    gos = gu + gv + gud + gvd + gel
    gos.show(xmin=-bnd, ymin=-bnd, xmax=bnd, ymax=bnd, aspect_ratio=1)
```
Rotate by $\theta$

\[ M = RP \]

\[
R = \begin{bmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{bmatrix}
\]

\[
P = \begin{bmatrix}
x \\
y
\end{bmatrix}
\]

\[
\begin{bmatrix}
\cos(\theta) x - \sin(\theta) y \\
\sin(\theta) x + \cos(\theta) y
\end{bmatrix} = \begin{bmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix}
\]

Does this make sense, given the geometry of the dot product?
Derivation of Rotation Matrix

\[ x_1 = r \cos(\theta) \quad x_2 = r \cos(\theta + \phi) \]
\[ y_1 = r \sin(\theta) \quad y_2 = r \sin(\theta + \phi) \]

Pronunciation Guide

\[ \theta \quad \text{theta} \]
\[ \varphi \quad \text{phi (fee)} \]
Trig. Identity:

\[
\begin{align*}
\cos(a + b) &= \cos(a)\cos(b) - \sin(a)\sin(b) \\
\sin(a + b) &= \sin(a)\cos(b) + \sin(b)\cos(a)
\end{align*}
\]

\[
\begin{align*}
x_2 &= r \cos(\theta + \phi) \\
x_2 &= r \cos(\theta)\cos(\phi) - r \sin(\theta)\sin(\phi) \\
x_2 &= x_1 \cos(\phi) - y_1 \sin(\phi)
\end{align*}
\]

Remember by Definition

\[
\begin{align*}
x_1 &= r \cos(\theta) \\
y_1 &= r \sin(\theta)
\end{align*}
\]

Y is left for you. Really, try it
Uniform Scaling

The first of several 2D canonical matrices...

\[ S = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}, \quad P = \begin{bmatrix} x \\ y \end{bmatrix}, \quad M = S \ P \]

\[
\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s \ x \\ s \ y \end{bmatrix}
\]

\[ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad M = s \ I \ P, \quad M = s \begin{bmatrix} x \\ y \end{bmatrix} \]
Non-uniform Scaling

\[ S = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} \quad P = \begin{bmatrix} x \\ y \end{bmatrix} \]

\[ M = S \cdot P = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s_1 x \\ s_2 y \end{bmatrix} \]

*Note orientation shift in line*
Flip an Axis...

\[
\begin{bmatrix}
x \\ -y
\end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]

What does this do to appearance of objects?
Flip the Horizontal

March 23, 2012

Enantiomorph

Look in the mirror. What do you see? A reflection? Nonsense! A reader of Uncommon Parlanse observes an enantiomorph: the fancy-pants term for a mirror image. Enantiomorphism also crops up in the field of chemistry where it refers to crystals that are structurally mirror images of each other. Etymology: from Ancient Greek ἐναντίος or enantios (opposite) + μορφή or morphē (form).

“The cardinal looked himself in the eye and curled his lip into a sneer. In the mirror his enantiomorph exhibited the same self-disgust and followed suit.”

\[
\begin{bmatrix}
  x_2 \\
  y_2
\end{bmatrix} = \begin{bmatrix}
  -1 & 0 \\
  0 & 1
\end{bmatrix} \begin{bmatrix}
  x_1 \\
  y_1
\end{bmatrix}
\]
Swap Axes

\[
\begin{bmatrix}
  y \\
  x
\end{bmatrix} = \begin{bmatrix}
  0 & 1 \\
  1 & 0
\end{bmatrix} \begin{bmatrix}
  x \\
  y
\end{bmatrix}
\]

Why would you do this?
Translation

\[ P(x, y) = x, y \]

\[ P(u, v) = u, v = x, y + a, b \]

I am intentionally drawing the alternative geometry, i.e. move the origin not the point.

Addition, not multiplication!

Plug in some values and draw yourself some pictures.
Canonical Transformations

\[ Rotate = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \]

\[ Flip = \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix} \]

\[ Scale = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} \]

\[ Translate = \begin{bmatrix} t_x \\ t_y \end{bmatrix} \]

\text{Add, don’t multiply}
Composition ...

To apply transformation $A$ to point $p$, and then transform the result by transformation $B$:

$$p' = (B \ A) \ p = B \ (A \ p)$$

*Question*: why is this important?
Problem: Translation

Unfortunately, we often need to translate points, and translation is matrix addition, not multiplication.

We need some way to make translation into a matrix multiplication operation, so that all transformations can be composed...
Solution:
Homogeneous Coordinates

In homogeneous coordinates, a two-dimensional point is represented as a vector of length 3.

In homogeneous coordinates, a three-dimensional point is represented as a vector of length 4.

In general, homogeneous coordinates represent an N-dimensional point with a vector of length N+1.
Homogeneous Coordinates (cont.)

In particular, the 2D point \((x, y)\) is:

\[
\begin{pmatrix}
  x \\
  y \\
  1
\end{pmatrix}
= \begin{pmatrix}
  2x \\
  2y \\
  2
\end{pmatrix}
= \begin{pmatrix}
  nx \\
  ny \\
  n
\end{pmatrix}
\]

For any \(n \neq 0\)

**Question:** what is the last coordinate (conceptually)?
Homogeneous Coordinates (cont...)

• Note that homogeneous coordinates are non-unique, but

• Translation in homogeneous coordinates is multiplication:

\[
\begin{pmatrix}
    x + t_x \\
    y + t_y \\
    1
\end{pmatrix} =
\begin{pmatrix}
    1 & 0 & t_x \\
    0 & 1 & t_y \\
    0 & 0 & 1
\end{pmatrix} \cdot
\begin{pmatrix}
    x \\
    y \\
    1
\end{pmatrix}
\]
Canonical Homogeneous Matrices

• **2D Rotation looks pretty much the same:**

\[
\begin{bmatrix}
x_2 \\
y_2 \\
w_2
\end{bmatrix} = \begin{bmatrix}
\cos(\theta) & -\sin(\theta) & 0 \\
\sin(\theta) & \cos(\theta) & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
y_1 \\
w_1
\end{bmatrix}
\]

• **As does 2D non-uniform scaling:**

\[
\begin{bmatrix}
x_2 \\
y_2 \\
w_2
\end{bmatrix} = \begin{bmatrix}
s_x & 0 & 0 \\
0 & s_y & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
y_1 \\
w_1
\end{bmatrix}
\]
Rotate about a Point

- Rotate the Cat’s Head about its Nose
  1. Translate the Nose to the Origin
  2. Rotate by the desired amount
  3. Invert the translation
Rotation about a Point (II)

• Translate to origin
  Note the negations: we want to bring \((t_x, t_y)\) to the origin, so subtract \(t_x, t_y\).

\[
M_T = \begin{bmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{bmatrix}
\]

• Rotate about origin
  What was the point \((t_x, t_y)\) is now at the origin.

\[
M_R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

• Translate back
  Finally, what started as \((t_x, t_y)\) is again \((t_x, t_y)\).

\[
M_{T^{-1}} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}
\]
Rotation about a Point (III)

\[ M = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{bmatrix} \]

- Think about order!
- Operations right before those on left.
- Therefore, read from right to left.
Rotation about a Point (IV)

• Reminder, in matrix multiplication:

\[ AB \neq BA \]

• The equation to rotate a matrix of points P around \((t_x, t_y)\) is:

\[ P' = T^{-1}RTP \]
Rotation about a Point (V)

- Compose the three transformations.

\[ P' = \left(T^{-1}RT\right)P \]

\[
P' = \begin{bmatrix}
\cos(\theta) & -\sin(\theta) & \sin(\theta)t_y + (1 - \cos(\theta))t_x \\
\sin(\theta) & \cos(\theta) & -\sin(\theta)t_x + (1 - \cos(\theta))t_y \\
0 & 0 & 1
\end{bmatrix} P
\]
Scaling About Point P

- Scaling also operates relative to the origin.
- To make an object bigger without moving it:
  - Translate origin to object centroid.
  - Apply scaling.
  - Invert the translation.

\[
M_4 = \begin{bmatrix}
1 & 0 & t_x \\
0 & 1 & t_y \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
s_x & 0 & 0 \\
0 & s_y & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & -t_x \\
0 & 1 & -t_y \\
0 & 0 & 1 \\
\end{bmatrix}
= \begin{bmatrix}
s_x & 0 & -s_x t_x + t_x \\
0 & s_y & -s_y t_y + t_y \\
0 & 0 & 1 \\
\end{bmatrix}
\]
One More Transform - Shear

*Shearing in the X dimension*

- Metaphor - Wind Blows Figure.

- Basic Matrix Form.

- Can X-Shear do This?
Shear (cont.)

Shearing in the Y dimension

- Metaphor - Same thing, other direction.

- Basic Matrix Form.

\[
\begin{pmatrix}
1 & 0 & 0 \\
sh_y & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

- Note: Parallel Lines Stay Parallel
Use Notebook 3 to develop an intuition as well as mechanical understanding!
The End