Lecture 16: Perspective Projection

October 18, 2018
Upcoming Assignments

Details will follow, but here is the overall programming arc from now to the end of the semester.

• Programming Assignment 3
  – Spheres, Specular Highlights and Shadows

• Programming Assignment 4
  – Reflections (recursion) and Texture

• Programming Assignment 5
  – Refraction, Scene Complexity, and Wow!
3D Viewing as Virtual Camera

To take a picture with a camera, or to render an image with computer graphics, we need to:

1. Position the camera/viewpoint in 3D space
2. Orient the camera/viewpoint in 3D space
3. Focus camera – *ray trace thin lens*
4. Crop image to the aperture/window
5. Project scene onto the image plane
Perspective ...
Orthographic Projection

If not for the fog, you could see forever ... and nothing ever would look smaller.
Orthographic / Perspective
Think About Rays
No! Technical programs, including for example Maple, often favor orthographic projection.
Math: Orthographic Projection

• Simply drop a dimension.

\[
\begin{bmatrix}
u \\
v \\
0 \\
1
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix}
\]

• Think of a bug hitting a windshield.

• No more z axis!
  – no more bug

Photo by Brian, Jeff Booth site
www.jeffbooth.net (creative common License)
Perspective Projection

• Light rays pass through the focal point.
  – a.k.a. Eye, principal reference point, or PRP.
• The image plane is an infinite plane in front of (or behind) the focal point.
• Images are formed by rays of light passing through the image plane.
• Common convention:
  – Image points are (u,v)
  – World points are (x,y,z)
Why “Pinhole” Camera?

• Because you can build a camera that exactly fits this description:
  – Create a fully-enclosed black box
    • So that no light enters
  – Put a piece of film inside it, facing front
  – Punch a pin-hole in the front face of the box

• What doesn’t this camera have?
• What is this camera’s depth-of-field?
• Why don’t we build cameras this way?
History

• The Camera Obscura - see Wikipedia


• Pre-dates photographic cameras.
  – Theory: Mo-Ti (China, 470-390 BC)
  – Practice: Abu Ali Al-Hasan Ibn al-Haitham (~1000 AD)
  – Western Painting: Johannes Vermeer (~1660 AD)
Pinhole Projection
Flip the Bear in the Box
Human Eye - 4 year old view

Drawing by Bryce Beveridge in 2006
Room Obscura
Perspective Projection

• Where we place the origin matters
• How we handle z values matters
• Form #1:  
  – Origin at focal point, z values constant
• Form #2:  
  – Origin at image center, z values are zero
• Form #3: (next lecture)  
  – Origin at focal point, z proportional to depth
The key to perspective projection is that all light rays meet at the PRP (E, focal point).

Notice that we are looking down the Z axis, with the origin at the focal point and the image plane at $z = d$. 

Perspective Projection Form #1
By similar triangles:

**horizontal**

\[
\frac{P_u}{d} = \frac{P_x}{P_z}
\]

\[
P_u = \frac{P_x d}{P_z}
\]

\[
P_u = P_x \frac{d}{P_z}
\]

**vertical**

\[
\frac{P_v}{d} = \frac{P_y}{P_z}
\]

\[
P_v = \frac{P_y d}{P_z}
\]

\[
P_v = P_y \frac{d}{P_z}
\]
Problem: division of one variable by another is a non-linear operation.

Solution: homogeneous coordinates!

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1/d & 0 \\
\end{pmatrix}
\]
Perspective Matrix (II)

\[
\begin{bmatrix}
 u \\
 v \\
 d \\
 1
\end{bmatrix} =
\begin{bmatrix}
 x \\
 y \\
 z \\
 d
\end{bmatrix} =
\begin{bmatrix}
 1 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 0 & 0 & 1/d & 0
\end{bmatrix}
\begin{bmatrix}
 x \\
 y \\
 z \\
 1
\end{bmatrix}
\]

Point in (u,v) coordinates

Point in Non-normalized Homogeneous coordinates

Projection Matrix times a Point

Normalized
What happens to Z?

• What happens to the Z dimension?

\[
\begin{bmatrix}
    u \\
    v \\
    d \\
    1
\end{bmatrix}
= 
\begin{bmatrix}
    x \\
    y \\
    z \\
    1
\end{bmatrix}
= 
\begin{bmatrix}
    1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 1/d & 0
\end{bmatrix}
\begin{bmatrix}
    x \\
    y \\
    z \\
    1
\end{bmatrix}
\]

• The Z dimension projects to d. Why?
• Because (u, v, d) is a 3D point on the image plane located at z = d!
Perspective Projection Form #2

\[ \frac{v}{d} = \frac{P_y}{d + P_z} \]

\[ v = P_y \left( \frac{d}{d + P_z} \right) \]
Leading to the following

\[
\begin{bmatrix}
  x \left( \frac{d}{d+z} \right) \\
  y \left( \frac{d}{d+z} \right) \\
  0 \\
  1
\end{bmatrix}
= \begin{bmatrix}
  x \\
  y \\
  0 \\
  \frac{z+d}{d}
\end{bmatrix}
= \begin{bmatrix}
  x \\
  y \\
  0 \\
  \frac{z}{d} + 1
\end{bmatrix}
= \begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & \frac{1}{d} & 1
\end{bmatrix}\begin{bmatrix}
  x \\
  y \\
  z \\
  1
\end{bmatrix}
\]

- Now look at what happens to depth.
- Contrast this with previous version.
Let distance $d$ go to infinity.

**Formulation #1**

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & \frac{1}{d} & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix}
= 
\begin{bmatrix}
x \\
y \\
z \\
\frac{1}{d}
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & \frac{1}{d} & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
\frac{1}{d}
\end{bmatrix}
= 
\begin{bmatrix}
\frac{dx}{z} \\
\frac{dy}{z} \\
\frac{dz}{d} \\
1
\end{bmatrix}
\]

**Formulation #2**

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{d} & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
1
\end{bmatrix}
= 
\begin{bmatrix}
x \\
y \\
z \\
0
\end{bmatrix}
= 
\begin{bmatrix}
\frac{x}{1 + \frac{z}{d}} \\
\frac{y}{1 + \frac{z}{d}} \\
\frac{z}{1 + \frac{z}{d}} \\
0
\end{bmatrix}
\]

Recall formulation #2 when considering how projection changes with increased focal length.