

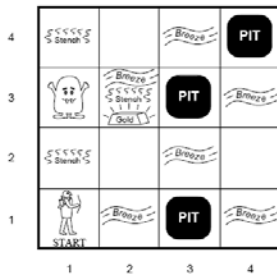
## Propositional Logic (continued)

Lecture #12  
10/02/08

## Announcements

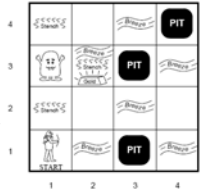
- Programming assignment #1 was due at noon
- Writing assignment #2 (part 1) is due Tuesday, Oct. 7<sup>th</sup>.
- Possible midterm dates:
  - Tuesday, Oct. 14<sup>h</sup>
  - Thursdayday, Oct. 16<sup>th</sup>
- Read Chapter 8 for Tuesday

## The Wumpus World



## The Wumpus World

- Performance measure
  - gold: +1000, death: -1000
  - -1 per step, -10 for using the arrow
- Environment
  - Squares adjacent to wumpus: smelly
  - Squares adjacent to pit: breezy
  - Glitter iff gold is in the same square
  - Shooting kills wumpus if you are facing it
  - Shooting uses up the only arrow
  - Grabbing picks up gold if in same square
  - Square [1,1] is empty
- Sensors: Stench, Breeze, Glitter, Bump
- Actuators: Left turn, Right turn, Forward, Grab, Release, Shoot



## Wumpus Logic

- Let  $P_{x,y}$  be the proposition that square  $(x,y)$  has a pit.
  - Note that there are exactly 16 of these distinct symbols
- Let  $W_{x,y}$  be the proposition that the wumpus is in square  $(x,y)$ 
  - Again, 16 distinct symbols.
- Let  $B_{x,y}$  be the proposition that a breeze is felt in square  $(x,y)$
- Let  $S_{x,y}$  be the proposition that a stench is detected in square  $(x,y)$
- A system with a total of 64 propositional symbols

## Wumpus Axioms

- $\neg W_{1,1}$ 
  - The wumpus is not in square 1,1
- $\neg P_{1,1}$ 
  - Square 1,1 does not contain a pit
- $(W_{1,1} \vee W_{1,2} \vee \dots \vee W_{4,3} \vee W_{4,4})$ 
  - A wumpus exists
- $W_{1,1} \Rightarrow (\neg W_{1,2} \wedge \neg W_{1,3} \wedge \dots \wedge \neg W_{4,3} \wedge \neg W_{4,4})$ 
  - If the wumpus is somewhere, its nowhere else
  - There are 16 of these rules...

## Wumpus Axioms (II)

- $S_{2,2} \Leftrightarrow (W_{1,2} \vee W_{3,2} \vee W_{2,3} \vee W_{2,1})$ 
  - If there is a stench, the wumpus is in a neighboring cell
  - Once again, there are 16 of these rules
  - Boundary cases involve fewer disjunctions
- $B_{2,2} \Leftrightarrow (P_{1,2} \vee P_{3,2} \vee P_{2,3} \vee P_{2,1})$ 
  - If there is a breeze, a neighboring cell has a pit
  - 16 rules and boundary cases...

## Wumpus Logic (II)

- If sensors in square [1,1] detect neither breeze nor smell, what do we know?

Precedent	Rule	Consequent
$\neg B_{1,1},$ $B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$	Modus Tollens	$\neg(P_{1,2} \vee P_{2,1})$
$\neg(P_{1,2} \vee P_{2,1})$	DeMorgans	$(\neg P_{1,2} \wedge \neg P_{2,1})$
$(\neg P_{1,2} \wedge \neg P_{2,1})$	Simplification	$\neg P_{1,2}$
$(\neg P_{1,2} \wedge \neg P_{2,1})$	Simplification	$\neg P_{2,1}$

## Conjunctive Normal Form

- A sentence is in CNF iff it is a conjunction of disjunctions of propositions and negated propositions
- Example:  
 $\neg(A \vee B \vee \neg C) \wedge (\neg A \vee C) \wedge B$

## Converting to CNF

- Every sentence can be converted to CNF, by sequentially eliminating all other symbols...

  1. Replace  $\alpha \Leftrightarrow \beta$  with  $(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$
  2. Replace  $\alpha \Rightarrow \beta$  with  $(\neg \alpha \vee \beta)$
  3. Move  $\neg$  "inward"
    1. Replace  $\neg(\neg \alpha)$  with  $\alpha$
    2. Replace  $\neg(\alpha \wedge \beta)$  with  $(\neg \alpha \vee \neg \beta)$
    3. Replace  $\neg(\alpha \vee \beta)$  with  $(\neg \alpha \wedge \neg \beta)$
  4. Replace  $(\alpha \vee (\beta \wedge \gamma))$  with  $(\alpha \vee \beta) \wedge (\alpha \vee \gamma)$

## Converting to CNF (II)

- While converting expressions, note that
  - $((\alpha \vee \beta) \vee \gamma)$  is equivalent to  $(\alpha \vee \beta \vee \gamma)$
  - $((\alpha \wedge \beta) \wedge \gamma)$  is equivalent to  $(\alpha \wedge \beta \wedge \gamma)$
- Why does this algorithm work?
  - Because  $\Rightarrow$  and  $\Leftrightarrow$  are eliminated
  - Because  $\neg$  is always directly attached to simply propositions
  - Because what is left must be  $\wedge$ 's and  $\vee$ 's, and the can be distributed over to make CNFs

## Converting to CNF (Example)

- Convert:  $(A \Rightarrow B) \vee (B \Leftrightarrow C)$
- $(A \Rightarrow B) \vee ((B \Rightarrow C) \wedge (C \Rightarrow B))$
- $(\neg A \vee B) \vee ((\neg B \vee C) \wedge (\neg C \vee B))$
- $((\neg A \vee B) \vee (\neg B \vee C)) \wedge ((\neg A \vee B) \vee (\neg C \vee B))$
- $(\neg A \vee (B \vee \neg B) \vee C) \wedge (\neg A \vee B \vee \neg C)$
- Question: are the first and last lines really equivalent?

## Double Checking

A	B	C	$(A \Rightarrow B) \vee (B \Leftrightarrow C)$	$(\neg A \vee B \vee \neg C)$
T	T	T	T	T
T	T	⊥	T	T
T	⊥	T	⊥	⊥
T	⊥	⊥	T	T
⊥	T	T	T	T
⊥	T	⊥	T	T
⊥	⊥	T	T	T
⊥	⊥	⊥	T	T

## Resolution

- CNF enables an inference rule called *resolution*:
  - $(\alpha \vee \beta) \wedge (\neg \beta \vee \gamma) \vdash (\alpha \vee \gamma)$
- In general,
  - If two disjunctive clauses are true, and
  - One clause contains a term that is the negation of a term in the other
  - Then the disjunction of all the *other* terms is true
  - Also, if the result contains the same term twice, you may drop one
    - E.g.  $(A \vee A \vee \dots) \vdash (A \vee \dots)$

## Resolution & Contradictions

- What happens if a CNF expression contains a contradiction?
  - Example:  $(\neg A \vee B) \wedge (A) \wedge (\neg B)$ 
    - Yes, this in CNF form...
- Resolving the first & last term produces
  - $(\neg A) \wedge (A)$
- Another step of resolution produces
  - $\{\}$

## Resolution & Contradictions (II)

- Producing an empty clause signifies that it can't be satisfied – nothing must be true.
- Moreover, *any* set of contradictory disjunctive clauses will reduce to  $\{\}$  under repeated application of resolution.
- And *only* contradictory clauses lead to  $\{\}$

## Automated Theorem Proving

- How do we automate the inference process?
  - Step 1: assume the negation of the consequent and add it to the axioms
  - Step 2: convert all axioms to CNF
    - i.e. a collection of disjunctive clauses
  - Step 3: Repeatedly apply resolution to remove terms until:
    - It produces an empty clause (contradiction), in which case the consequent is proven, or
    - No more terms can be resolved, in which case the consequent cannot be proven

## Example

- Problem: prove the following
  - $(A \Leftrightarrow B) \wedge (A \Rightarrow C) \wedge B \vdash C$
- Step 1: negate the consequent
  - $(A \Leftrightarrow B) \wedge (A \Rightarrow C) \wedge B \wedge \neg C$
- Step 2: convert to CNF
  - $(A \Rightarrow B) \wedge (B \Rightarrow A) \wedge (A \Rightarrow C) \wedge B \wedge \neg C$
  - $(\neg A \vee B) \wedge (\neg B \vee A) \wedge (\neg A \vee C) \wedge B \wedge \neg C$

### Example (cont.)

- Now, apply resolution to  
 $(\neg A \vee B) \wedge (\neg B \vee A) \wedge (\neg A \vee C) \wedge (B) \wedge (\neg C)$   
–  $(\neg A \vee C) \wedge (\neg C) \vdash (\neg A)$
- Repeat:  
 $(\neg B \vee A) \wedge (\neg A) \vdash (\neg B)$
- Repeat:  
 $(\neg B) \wedge (B) \vdash \{\}$  : contradiction!

### Example (III)

- Or, alternatively, start with the first two clauses:  
 $(\neg A \vee B) \wedge (\neg B \vee A) \vdash (B) \wedge (\neg B)$   
 $(B) \wedge (\neg B) \vdash \{\}$

### What can resolution do?

- It can determine whether any sentence can be proven from a set of axioms.  
– It is *complete*.
- It cannot enumerate the set of all entailed sentences.  
– You can enumerate such an (infinite) list by enumerating all possible sentences and verifying each with resolution.