CS475 Parallel Programming

Shortest Paths

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Minimal Spanning Tree (MST)

- Spanning tree of an undirected graph G
  - A tree that is a sub-graph of G containing ALL vertices
- Minimal spanning tree of a weighted graph G
  - Spanning tree with minimal total weight
- G must be a connected graph
- Applications
  - Lowest cost set of roads connecting a set of towns
  - Shortest cable connecting a set of computers

Prim's Algorithm for MST

- Pick an arbitrary vertex
- Grow MST by choosing a new vertex v and edge e
  - such that they are guaranteed to be in the final, correct MST
  - Select least-cost (minimal) edge e(u,v) such that
    - u is already in MST
    - v is not in MST as yet
- Keep doing this until all vertices are in MST
- This is a GREEDY algorithm
  - a locally optimizing strategy leading to a global optimum

Properties of any tree hence MST

- Path between two nodes *a* and *b* in MST is unique
- Cycles in MST
  - there are no cycles
  - If *a* and *b* are non-adjacent, adding the edge (*a*,*b*) creates a cycle
  - Removing any edge on that cycle makes it a tree again

How greedy works for MST

- Consider each stage M with partial MST
  - Add the least-cost edge to M to obtain the next stage M'
  - The resulting MST will be minimal.
- Exchange Argument
  - Suppose we can create an MST by not taking the minimal cost edge
  - Call the minimal edge e, and the non-minimal edge taken e'
  - Build the rest of the spanning tree
  - We can now make a lower cost spanning tree by removing e' and adding e
  - Hence the spanning tree with e' in it was not minimal

Prim's Algorithm Code Structure

// Pick vertex r and initialize V_t, E_t, d and e

$V_t = \{ r \}$; $E_t = \{ \}$ // MST in construction

d[r] = 0 ; // d is a heap

$\forall$ v $\in$ V if ((r,v) $\in$ E) { d[v] = w(r,v) ; e[v] = r ; }

else d[v] = ∞ ;

// grow the MST

while V_t != V

Select vertex u from V-V_t with minimal d[u] ;

$V_t = V_t + u$; $E_t = E_t + (u,e[u])$;

// update d and e

$\forall$ v $\in$ V-V_t if (w(u,v) < d[v]) d[v]=w(u,v);e[v]=u

Complexity, parallelization of Prim

- while-loop executed n-1 times
- Loop-body O(n) if arrays are used
- Sequential complexity: O(mlogn)
- while-loop is sequential in nature, because of the data dependencies in V_t, E_t, d and e
- ∀ loops can be parallelized

Parallel Implementation of Prim

- Data distribution
  - Each PE has data for n/p vertices
  - Adjacency matrix A is block striped (column-wise)
  - d and e block striped
- PEs compute a local minimum u_l
- Local minima accumulate to give global minimum in PE_0
- PE_0 broadcasts global minimum u_g
- PE owning u_g updates V_t ,E_t
- All PEs update their partition of d and e using their columns of A

Single Source Shortest Path - SSSP

- Given a vertex s and weighted graph G, find the shortest distances from s to each vertex
- Dijkstra's algorithm (very much like Prim)

$V_t = \{ s \}$

$\forall\ v \in V\text{-}Vt$ if $((s,v) \in E)$ $l[v] = w(s,v)$; else $l[v] = \infty$;

while V_t != V

  Select vertex u from $V\text{-}V_t$ with minimal $l[u]$

  $V_t = V_t + u$

  $\forall\ v \in V\text{-}V_t$ $l[v] = \min(l[v], l[u]+w(u,v))$

Parallel SSSP

- Very similar to Prim
- Data distribution
  - n/p vertices per PE
  - Column distribute A
  - block distribute l
- Find minimal u_l locally
- Accumulate to obtain global minimum u_g
- Broadcast global minimum u_g
- Every PE updates its l block using its column-block of A

All pairs shortest paths - APSP

- Find length of shortest path between all vertex pairs
  - n*n distance matrix D: Dij is shortest distance for $v_i \rightarrow v_j$
- Algorithm: Floyd's APSP

Dynamic Programming approach

- Formulate the problem in a recursive fashion
- Reverse this formulation to create a BOTTOM UP solution
  - Use solutions for smaller problems to create solutions for larger ones
- There can be multiple recursive formulations
  - Recurrence on path length (Matrix Multiply formulation)
  - Recurrence on node set (Floyd's algorithm)

Floyd's APSP

- Terms used
  - node (sub)set $V_k = \{v_1, v_2, \ldots\ldots, v_k\}$
  - P_{ij}^k = minimal length path from v_i to v_j passing through nodes in V_k
  - d_{ij}^k = length of the path P_{ij}^k
- Recursion: based on node sets
  - Two possibilities: $\mathbf{v_k}$ in $\mathbf{P_{ij}^k}$ or not
    - v_k not in P_{ij}^k: $P_{ij}^k = P_{ij}^{k-1}$ and $d_{ij}^k = d_{ij}^{k-1}$
    - v_k in P_{ij}^k: $P_{ij}^k = P_{ik}^{k-1} + P_{kj}^{k-1}$ and $d_{ij}^k = d_{ik}^{k-1} + d_{kj}^{k-1}$
- $d_{ij}^k = \min(d_{ij}^{k-1}, d_{ik}^{k-1} + d_{kj}^{k-1})$ for $k > 0$

  $= w(v_i, v_j)$ for $k = 0$
- Solution $D = D^{\mathbf{n}}$

Floyd's APSP (Sequential)

$D^0 = A$

for k = 1 to n

    for i = 1 to n

        for j = 1 to n

            $D_{ij}^k = \min(D_{ij}^{k-1}, D_{ik}^{k-1} + D_{kj}^{k-1})$

$O(n^3)$ sequential time complexity

$O(n^2)$ space complexity

Floyd Parallel

- Mesh checkerboard partitioning
- Iteration k: Broadcast k-th row and k-th column of D

for k = 1 to n

each PE having a segment of row k of D^{k-1} broadcast it in its column

each PE having a segment of column k of D^{k-1} broadcasts it in its row

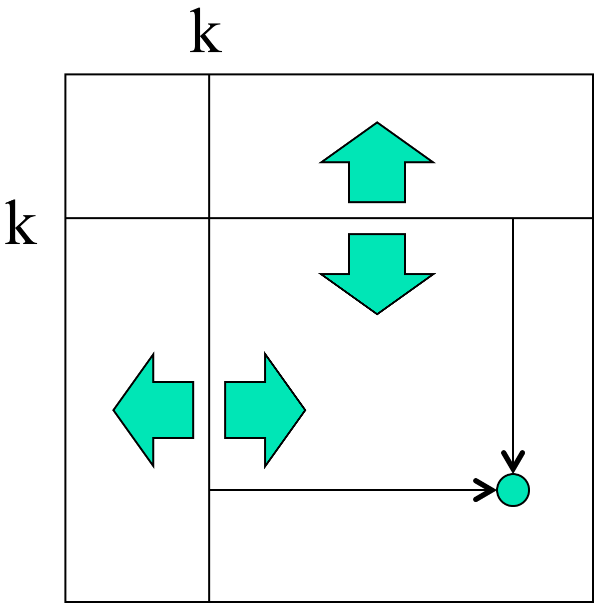
each PE waits to receive the needed segments of D^{k-1}

each PE computes its part of D^k

- Note that this algorithm can be pipelined like Gaussian elimination or LUD

Floyd Parallel: update in place

- In the kth iteration
- D_{ik} and D_{kj} are broadcast and do not change
- other elements D_{ij} depend on D_{ik} and D_{kj} and themselves (no other elements depend on D_{ij})



So there are no data hazards, and all elements can be updated in place

Transitive Closure

- Given: graph G=(V,E)

  Transitive closure: G*=(V,E*)
  - E* = {(v_1,v_2) | ∃ path from v_1 to v_2 in G}
- Connectivity matrix A*
  - $A_{ij}^* = 1$ if (v_i,v_j) in E* or i = j

    = 0 otherwise
- Use Floyd
  - replacing *min* by *or* and *sum* by *and*