Geometric Transformations, 8DOF Transform

Lecture #5

Monday February 4, 2019

Colorado State University

Classes of Image Transformations

- Rigid transformations
  - Combine rotation and translation
  - Preserve relative distances and angles
  - 3 Degrees of freedom
- Similarity transformations
  - Add scaling to rotation and translation
  - Preserves relative angles
  - 4 Degrees of freedom

Affine Transformations

- Any transform of the form:

$$\begin{vmatrix} u \\ v \\ 1 \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} x \\ y \\ 1 \end{vmatrix}$$

- 6 DOF
- Arbitrary combination of
  - translations
  - rotations
  - scales (uniform or non-uniform)
  - shears

Similarity vs. Affine Matrices

- Similarity : 4 DOF

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ -b & a & d \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Affine : 6 DOF

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Specifying Affine Transformations

- There are six unknowns in the matrix (*a* through *f*)
- If you specify one point in the source image and a corresponding point in the target image, that yields two equations:

$$u_i = ax_i + by_i + c$$
$$v_i = dx_i + ey_i + f$$

- So providing three point-to-point correspondences specifies an affine matrix

Perspective Transformations

- We can go beyond affine transformations.
- We can do any perspective transformation of a one 3D view of a plane to another view.
- Therefore, we can model an image as a plane in space, and project it onto any other image.
  - How does this differ from the perspective projection pipeline in CS410?

Perspective Matrix

$$\begin{bmatrix} u' \\ v' \\ w \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$u = {}^{u'}\!/_{w} \;,\; v = {}^{v'}\!/_{w}$$

- Why does element [3,3] = 1?
- How many points are needed to specify this matrix?

Solving for Perspective

- Four corresponding points produce eight equations, eight unknowns --- but we can't observe *w*

$$u_i = {u'_i}/{w_i} = \frac{ax_i + by_i + c}{gx_i + hy_i + 1}$$

$$v_i = {v'_i}/{w_i} = \frac{dx_i + ey_i + f}{gx_i + hy_i + 1}$$

Solving (cont.)

- Multiply to get rid of the fraction…

$$u_i\left(gx_i + hy_i + 1\right) = ax_i + by_i + c$$
$$v_i\left(gx_i + hy_i + 1\right) = dx_i + ey_i + f$$

- Now, remember that the *u's, v's, x's* & *y's* are known; group the unknown terms

$$u_i = ax_i + by_i + c - gx_iu_i - hy_iu_i$$
$$v_i = dx_i + ey_i + f - gx_iv_i - hy_iv_i$$

Solving (III)

- And express the result as a system of linear equations

$$\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 & -x_1u_1 & -y_1u_1 \\ 0 & 0 & 0 & x_1 & y_1 & 1 & -x_1v_1 & -y_1v_1 \\ x_2 & y_2 & 1 & 0 & 0 & 0 & -x_2u_2 & -y_2u_2 \\ 0 & 0 & 0 & x_2 & y_2 & 1 & -x_2v_2 & -y_2v_2 \\ x_3 & y_3 & 1 & 0 & 0 & 0 & -x_3u_3 & -y_3u_3 \\ 0 & 0 & 0 & x_3 & y_3 & 1 & -x_3v_3 & -y_3v_3 \\ x_4 & y_4 & 1 & 0 & 0 & 0 & -x_4u_4 & -y_4u_4 \\ 0 & 0 & 0 & x_4 & y_4 & 1 & -x_4v_4 & -y_4v_4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{bmatrix}$$

Solving (IV)

- Finally, invert the constant matrix and solve

$$\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 & -x_1u_1 & -y_1u_1 \\ 0 & 0 & 0 & x_1 & y_1 & 1 & -x_1v_1 & -y_1v_1 \\ x_2 & y_2 & 1 & 0 & 0 & 0 & -x_2u_2 & -y_2u_2 \\ 0 & 0 & 0 & x_2 & y_2 & 1 & -x_2v_2 & -y_2v_2 \\ x_3 & y_3 & 1 & 0 & 0 & 0 & -x_3u_3 & -y_3u_3 \\ 0 & 0 & 0 & x_3 & y_3 & 1 & -x_3v_3 & -y_3v_3 \\ x_4 & y_4 & 1 & 0 & 0 & 0 & -x_4u_4 & -y_4u_4 \\ 0 & 0 & 0 & x_4 & y_4 & 1 & -x_4v_4 & -y_4v_4 \end{bmatrix}^{-1} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{bmatrix}$$

Solving (V) : Questions

- Is there always a solution?
- Under what conditions is the matrix invertible?
- Is the solution always unique?

Perspective Image Transforms (Intuition)

- What does the following matrix do?

$$\begin{bmatrix} \sqrt{2} & -\sqrt{2} & 0 \\ \sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix Decomposition

$$\begin{bmatrix} \sqrt{2} & -\sqrt{2} & 0 \\ \sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

original

Scale by 2

Rotation by 45

- Note that such decompositions are:
  - not unique (why?)
  - difficult to intuit

More Intuitions

- What will the following matrix do?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

- More specifically, what will it do to the giraffe image?

Check Your Intuitions

- What's going on here?

More Intuition Checking

- Part of what you are seeing is a scale effect
  - positive terms in the bottom row create larger w values, and therefore smaller u, v values
- Something much weirder is also going on:
  - What happens when $y = x+1$?
  - How do you interpret this geometrically?
  - Isn't the perspective transform linear?
- So how do you select transformation matrices?

Perspective Transform of 2D Planes Continued.

- Recall the basic equation for the perspective transform

$$\begin{bmatrix} u' \\ v' \\ w \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$u = {}^{u'}\!/_{w} \;,\; v = {}^{v'}\!/_{w}$$

- The only practical way to specify an image transform is by providing four point correspondences

Computing Transformations

- Remember how to build a transformation from four point correspondences….

$$\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 & -x_1u_1 & -y_1u_1 \\ 0 & 0 & 0 & x_1 & y_1 & 1 & -x_1v_1 & -y_1v_1 \\ x_2 & y_2 & 1 & 0 & 0 & 0 & -x_2u_2 & -y_2u_2 \\ 0 & 0 & 0 & x_2 & y_2 & 1 & -x_2v_2 & -y_2v_2 \\ x_3 & y_3 & 1 & 0 & 0 & 0 & -x_3u_3 & -y_3u_3 \\ 0 & 0 & 0 & x_3 & y_3 & 1 & -x_3v_3 & -y_3v_3 \\ x_4 & y_4 & 1 & 0 & 0 & 0 & -x_4u_4 & -y_4u_4 \\ 0 & 0 & 0 & x_4 & y_4 & 1 & -x_4v_4 & -y_4v_4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{bmatrix}$$

Computing...

- So if we want the following mapping:

$$(0,0)\rightarrow(0,0),\ (0,144)\rightarrow(0,144),$$
$$(152,0)\rightarrow(152,50),\ (152,144)\rightarrow(152,94)$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 144 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 144 & 1 & 0 & -20736 \\ 152 & 0 & 1 & 0 & 0 & 0 & -23104 & 0 \\ 0 & 0 & 0 & 152 & 0 & 1 & -7600 & 0 \\ 152 & 144 & 1 & 0 & 0 & 0 & -23104 & -21888 \\ 0 & 0 & 0 & 152 & 144 & 1 & -14288 & -13536 \end{bmatrix}$$

…More Computing...

What does This say about x?

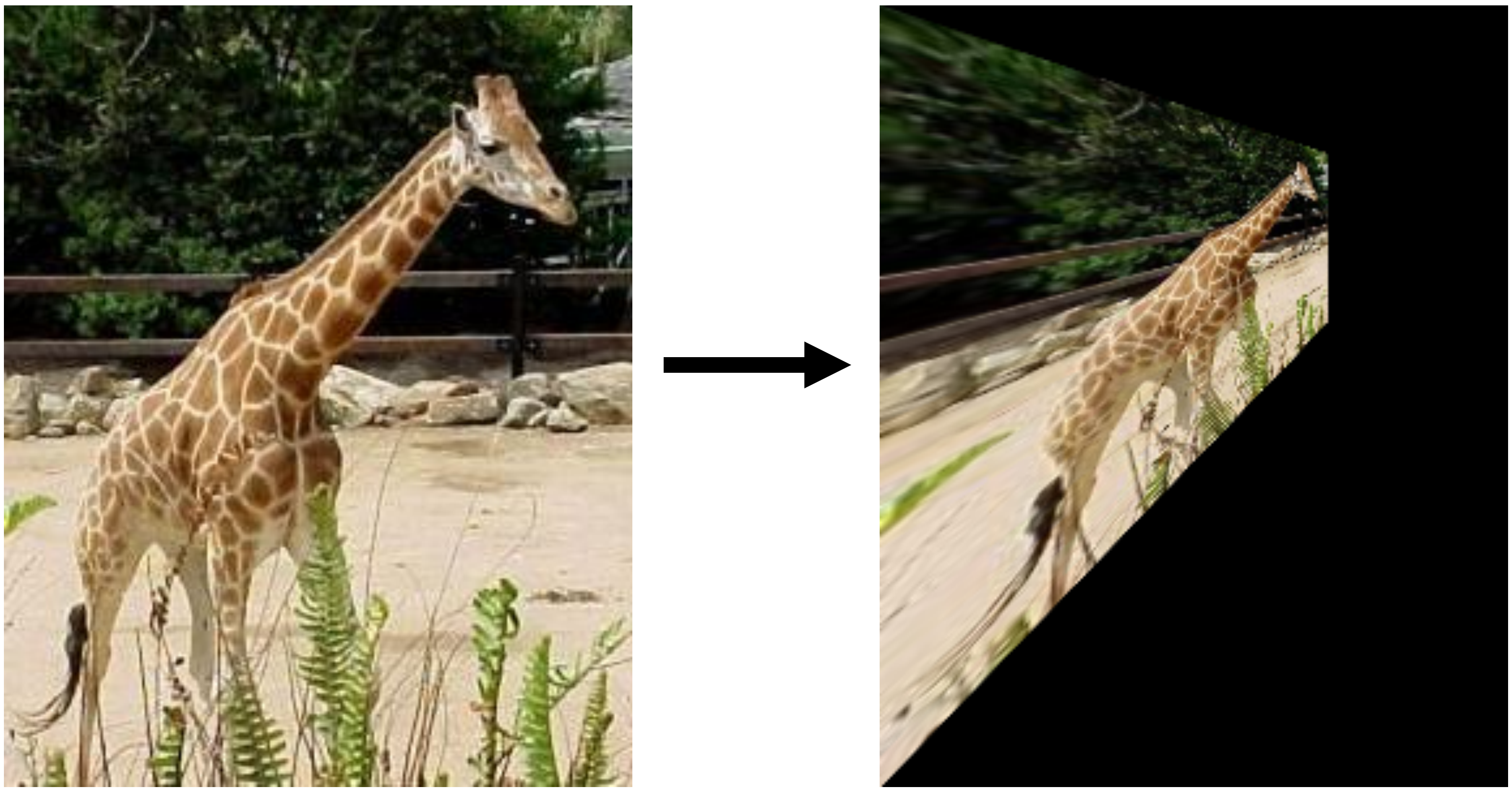
$$\underbrace{\begin{bmatrix} -.014 & -.023 & .007 & .023 & .014 & .022 & -.007 & -.022 \\ -.007 & 0 & .007 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -.002 & -.014 & .002 & .007 & .002 & .014 & -.002 & -.007 \\ -.007 & -.007 & .007 & .007 & .007 & 0 & -.007 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ .000 & -.000 & .000 & .000 & .000 & .000 & -.000 & .000 \\ -.000 & 0 & .000 & 0 & .000 & 0 & -.000 & 0 \end{bmatrix}}_{M^{-1}} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 144 \\ 152 \\ 50 \\ 152 \\ 94 \end{bmatrix}}_{\text{u\&v vector}} = \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{bmatrix} = \begin{bmatrix} 3.274 \\ 0 \\ 0 \\ 1.077 \\ 1 \\ 0 \\ .01497 \\ 0 \end{bmatrix}$$

Remember the earlier WLOG? $c = u_1, \ldots$

How does This alter it?

…yields

$$\begin{bmatrix} 3.274 & 0 & 0 \\ 1.077 & 1 & 0 \\ .01497 & 0 & 1 \end{bmatrix}$$



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