Introduction to Fourier Analysis – Part 1

CS 510
Lecture #6
February 6, 2019
Get used to changing representations!
(and this particular transformation is ubiquitous and important)
Why Fourier in Particular

• How much information can an image hold?
  – If I shrink it, then make it larger again, does it look the same?
  – If I enlarge it and then shrink it again does it look the same?
  – How about other transformations?

• How do I compare two images with different source resolutions?
Image Data as a Function

\[ g(x) = \text{Intensity} \]

Any repeating pattern can be built up from a sequence of sin/cosine waves.

Graphic adapted from http://www.revisemri.com/images/ft.gif
In the extreme, a square wave

Graphic from http://www.mechatronics.colostate.edu/figures/4-4.jpg
Fourier Analysis ≠ Magic

• Many textbooks make it obscure, but...

• Rewrite a function f(x) over a finite range.
  – In effect, we pretend it repeats
  – …and reconstruct it as a sum of sinusoids…

• For each sinusoid, we specify:
  – A frequency
  – An amplitude
  – A phase
The Sine Wave - Basics

- Amplitude
- Phase
- Period (Frequency = 1/Period)
The Sinusoid

\[ g(x) = a \cos(fx + \phi) \]

- **Amplitude**: \(a\)
- **Frequency**: \(f\)
- **Phase**: \(\phi\)
Simplifying Phase

• Phase – where wave crosses the x axis:
  – If it crosses at 0 and -\(\pi\), it’s a sine wave.
  – If it crosses at \(\pi/2\) and - \(\pi/2\), it a cosine wave.
  – In general, if it crosses at \(\phi\) and \(\phi + \pi\) radians, it has phase \(\phi - \pi/2\) (i.e., cosine, not sine)
    • \(\phi = 0\) \(\Rightarrow\) cosine wave
    • \(\phi = \pi/2\) \(\Rightarrow\) sine wave

• A wave with phase \(\phi\) can be expressed as:
  \[
  \cos(x + \phi) = \alpha \cos(x) + \beta \sin(x)
  \]
Writing Phase

• Repeating the punchline from previous slide.

• A sinusoid of arbitrary phase can be written as the sum of a sine & cosine:

\[ \cos(x + \phi) = \alpha \cos(x) + \beta \sin(x) \]
Phase (II)

Where:

\[ \phi = \tan^{-1}\left( \frac{\beta}{\alpha} \right) \]

\[ \cos(\theta + \phi) = \sqrt{\alpha^2 \cos^2(\theta) + \beta^2 \sin^2(\theta)} \]

\((\theta + \phi)\) still indicates that the curve has been shifted by \(\phi\) degrees.
Therefore...

\[ g(x) = a_1 \cos(f_1 x) + b_1 \sin(f_1 x) + a_2 \cos(f_2 x) + b_2 \sin(f_2 x) + a_3 \cos(f_3 x) + b_3 \sin(f_3 x) + \cdots \]

- Remaining problems
  - Now has twice as many terms (2 per frequency)
  - Must specify amplitude and frequency for each
Visualization – Nice Introduction

Another great resource ...
... I forgot to mention ...

Euler's formula

From Wikipedia, the free encyclopedia

This article is about Euler's formula in complex analysis. For Euler's formula in algebraic topology and polyhedral combinatorics see Euler characteristic.

Euler's formula, named after Leonhard Euler, is a mathematical formula in complex analysis that establishes the fundamental relationship between the trigonometric functions and the complex exponential function. Euler's formula states that, for any real number $x$,

$$e^{ix} = \cos x + i \sin x$$

where $e$ is the base of the natural logarithm, $i$ is the imaginary unit, and $\cos$ and $\sin$ are the trigonometric functions cosine and sine respectively, with the argument $x$ given in radians. This complex exponential function is sometimes denoted $\text{cis}(x)$ ("cosine plus $i$ sine"). The formula is still
Now, Per Frequency

• For any given frequency $f_1$, we must calculate the amplitude of the cosine and sine functions

• $g(x)$ is infinite ($x$ can be anything), so we have to fit the cosines and sines to all the data:

$$a_1 = \int_{-\infty}^{+\infty} f(x) \cos(f_1 x) \, dx \quad b_1 = \int_{-\infty}^{+\infty} f(x) \sin(f_1 x) \, dx$$
Representing Frequencies

• Our function \( g(x) \) is infinitely repeating
• Assume, WLOG, the unit of repetition is 1
  – So \( g(0+x) = g(1+x) = g(2+x) \ldots \)
• Only cosines & sines with periods that are integers make sense
  – Otherwise it wouldn’t repeat
• The cosine whose period is 1 is written as
  \[
  \cos(2\pi x)
  \]
Frequencies (cont).

• In general the frequency that repeats \(u\) times over the interval is written as

\[\cos(2\pi ux)\]

• Where \(u\) is any integer
So, to rewrite \( g(x) \)

- For every integer frequency \( u = 1, 2, 3, \ldots \)

\[
g(x) = a_1 \cos(2\pi x) + b_1 \sin(2\pi x) \\
+ a_2 \cos(2\pi 2x) + b_2 \sin(2\pi 2x) \\
+ a_3 \cos(2\pi 3x) + b_3 \sin(2\pi 3x) \\
+ a_4 \cos(2\pi 4x) + b_4 \sin(2\pi 4x) \\
+ \ldots
\]

\[
a_u = \int_{-\infty}^{\infty} g(x) \cos(2\pi ux) \, dx \\
b_u = \int_{-\infty}^{\infty} g(x) \sin(2\pi ux) \, dx
\]

Think about bounds
Fourier Transform

• Formally, the Fourier transform in 1D is:

\[ F(u) = \int_{-\infty}^{+\infty} f(x) \left[ \cos 2\pi ux - i \sin 2\pi ux \right] dx \]

Where:
- \( u \) is an integer in the range from 0 to \( \infty \)
- \( -i \) is used to create a 2D vector space
- \( F(u) = a_u + ib_u \)
The DC component

• What happens when $u = 0$?
  – $\cos(0) = 1$, $\sin(0) = 0$

• So

$$F(u = 0) = \int_{-\infty}^{+\infty} f(x)[\cos 2\pi u x - i \sin 2\pi u x] \, dx$$

  = $\int_{-\infty}^{+\infty} f(x) \, dx$

  – This is the average value (or “DC component”) of the function. For images, it is largely a function of lighting and camera gain.
Inverse Fourier Transform

• What if I have $F(u)$ for all $u$, and I want to recreate the original function $g(x)$?

• Well, sum it up for every $u$:

$$f(x) = \int_{-\infty}^{+\infty} F(u)[\cos(2\pi ux) + i \sin(2\pi ux)] \, du$$
Discrete Fourier Transform

• Problem: an image is not an analogue signal that we can integrate.
• Therefore for $0 \leq x < N$ and $0 \leq u < N/2$:

$$F(u) = \sum_{x=0}^{N-1} f(x) \left[ \cos\left(\frac{2\pi ux}{N}\right) - i \sin\left(\frac{2\pi ux}{N}\right) \right]$$

And the discrete inverse transform is:

$$f(x) = \frac{1}{N} \sum_{u=0}^{N-1} F(u) \left[ \cos\left(\frac{2\pi ux}{N}\right) + i \sin\left(\frac{2\pi ux}{N}\right) \right]$$
Discrete vs. Continuous

• Summation replaces integration

• Division by N (the number of discrete samples) makes the unit of repetition 1.

• For any signal (continuous or discrete)
  – G(x) is called the spatial domain
  – F(u) is called the frequency domain
Spatial vs Frequency

• Spatial domain representation size?
  – Given N samples, it is size N

• Frequency domain representation size?
  – A total of N/2 frequencies
  – A complex number (2 values) per frequency

• The DFT is invertible, so the two representations are equivalent:
  – Exact same information and same size
  – The DFT is O(n log n)