Introduction to Fourier Analysis – Part 1

CS 510

Lecture #6

February 6, 2019

Colorado State University

Why?

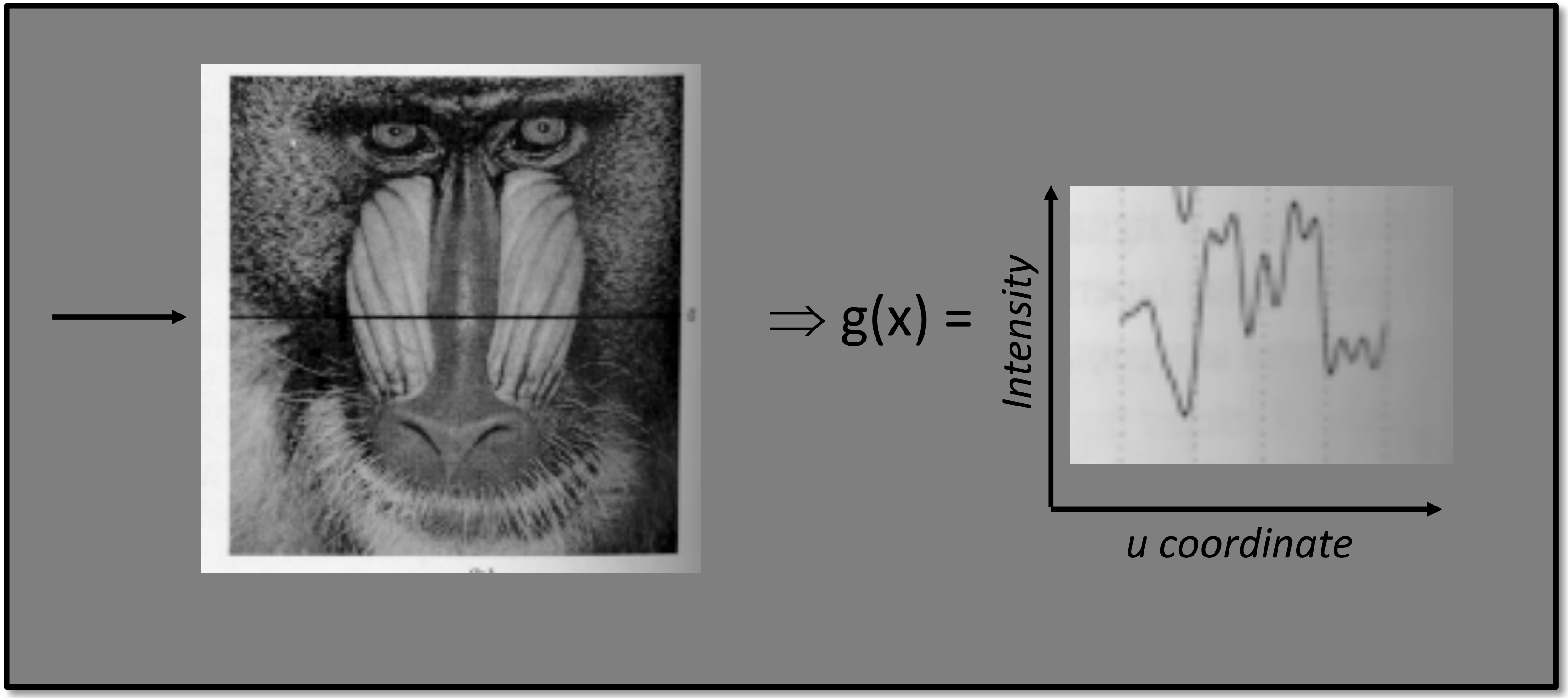
Get used to changing representations!

(and this particular transformation is ubiquitous and important)

Why Fourier in Particular

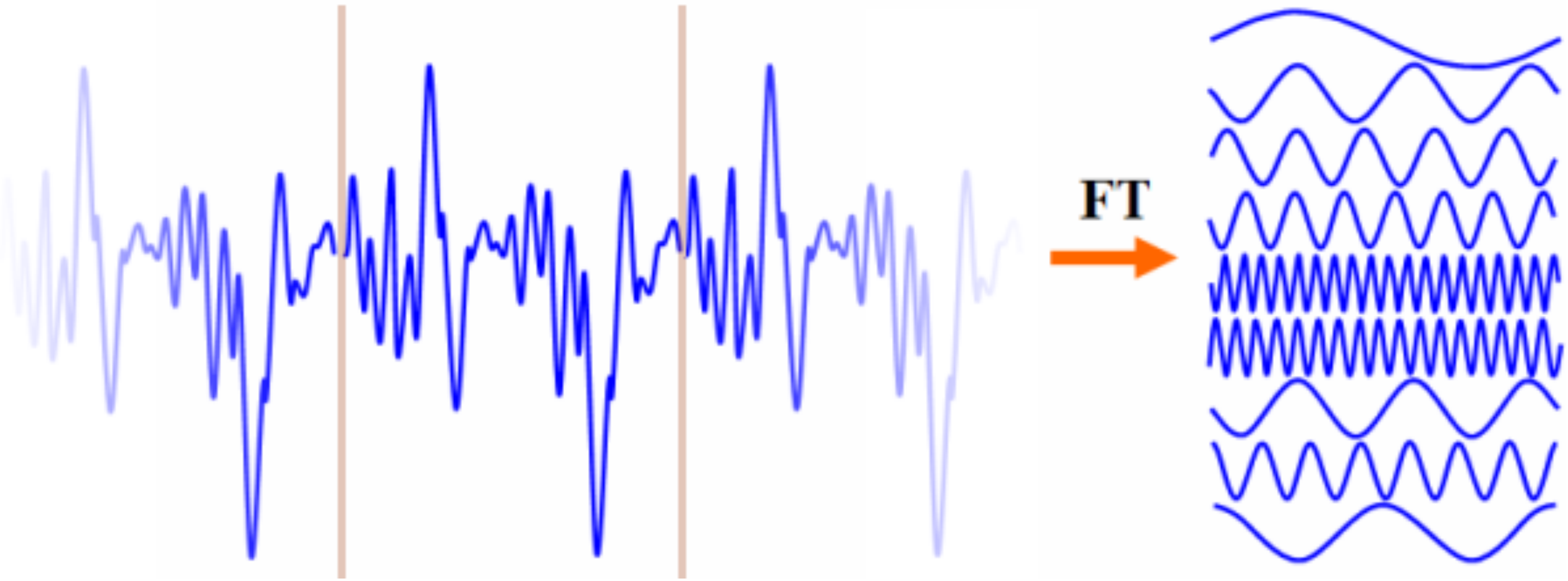
- How much information can an image hold?
  - If I shrink it, then make it larger again, does it look the same?
  - If I enlarge it and then shrink it again does it look the same?
  - How about other transformations?
- How do I compare two images with different source resolutions?

Image Data as a Function



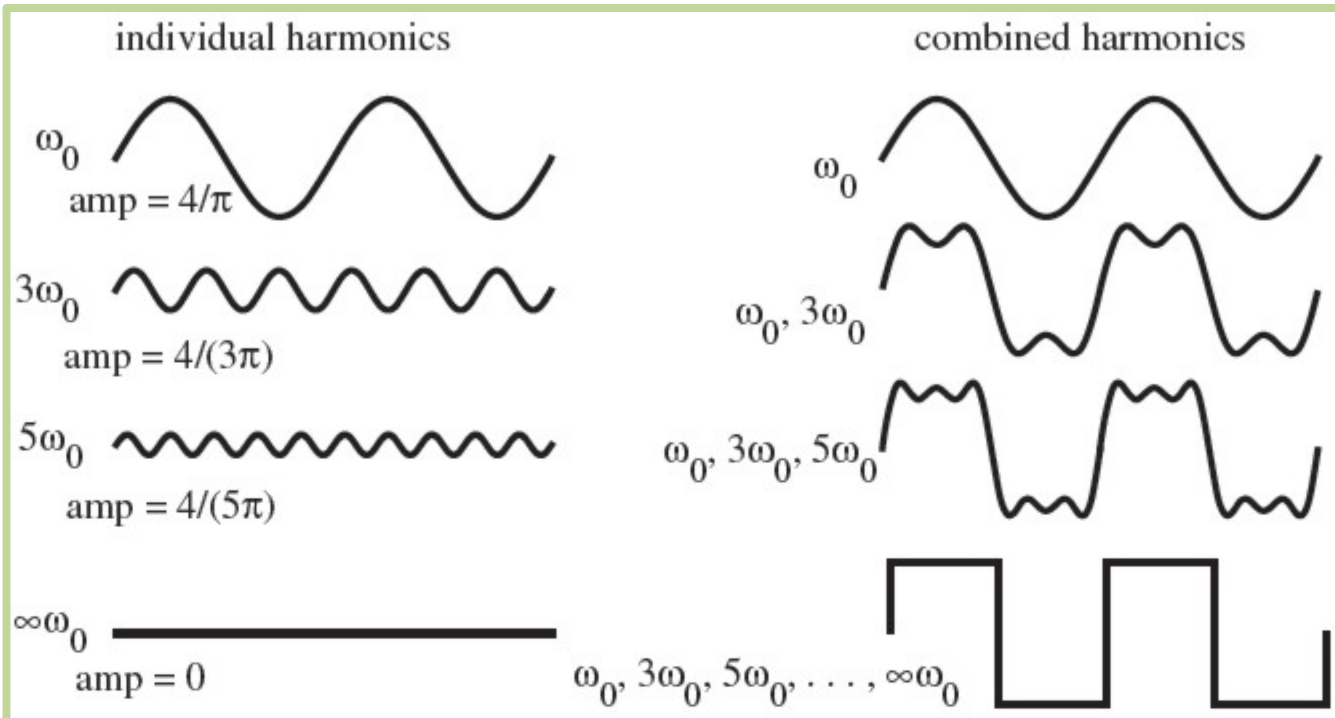
Graphic from *"Computer Graphics:Principles and Practice" by Foley, van Dam, Feiner & Hughes.*

Any repeating pattern can be built up from a sequence of sin/cosine waves.



Graphic adapted from *http://www.revisemri.com/images/ft.gif*

In the extreme, a square wave



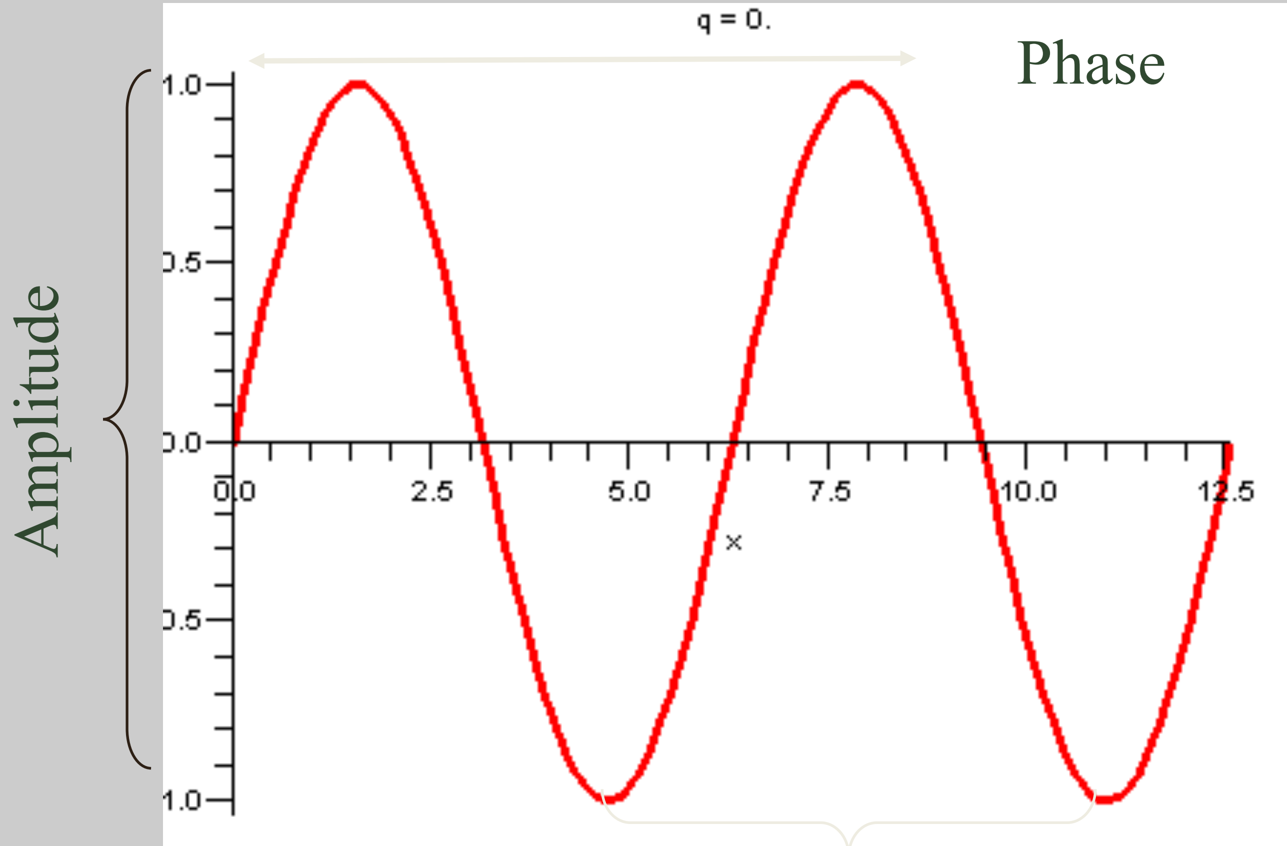
Graphic from *http://www.mechatronics.colostate.edu/figures/4-4.jpg*

Colorado State University

Fourier Analysis ≠ Magic

- Many textbooks make is obscure, but…
- Rewrite a function f(x) over a finite range.
  - In effect, we pretend it repeats
  - …and reconstruct it as a sum of sinusoids…
- For each sinusoid, we specify:
  - A frequency
  - An amplitude
  - A phase

The Sine Wave - Basics



Period (Frequency = 1/Period)

Colorado State University

The Sinusoid

Frequency

$$g(x) = \underline{a}\cos\left(\overline{f}x + \underline{\phi}\right)$$

Amplitude

Phase

Simplifying Phase

- Phase – where wave crosses the x axis:
  - If it crosses at 0 and $-\pi$, it's a sine wave.
  - If it crosses at $\pi/2$ and $-\pi/2$, it a cosine wave.
  - In general, if it crosses at ϕ and $\phi + \pi$ radians, it has phase $\phi - \pi/2$ (i.e., cosine, not sine)
    - $\phi = 0 \Rightarrow$ cosine wave
    - $\phi = \pi/2 \Rightarrow$ sine wave
- A wave with phase ϕ can be expressed as:

$$\cos(x+\phi) = \alpha\cos(x) + \beta\sin(x)$$

Writing Phase

- Repeating the punchline from previous slide.
- A sinusoid of arbitrary phase can be written as the sum of a sine & cosine:

$$\cos(x+\phi) = \alpha\cos(x) + \beta\sin(x)$$

Phase (II)

Where:

$$\phi = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$$

$$\cos(\theta + \phi) = \sqrt{\alpha^2 \cos^2(\theta) + \beta^2 \sin^2(\theta)}$$

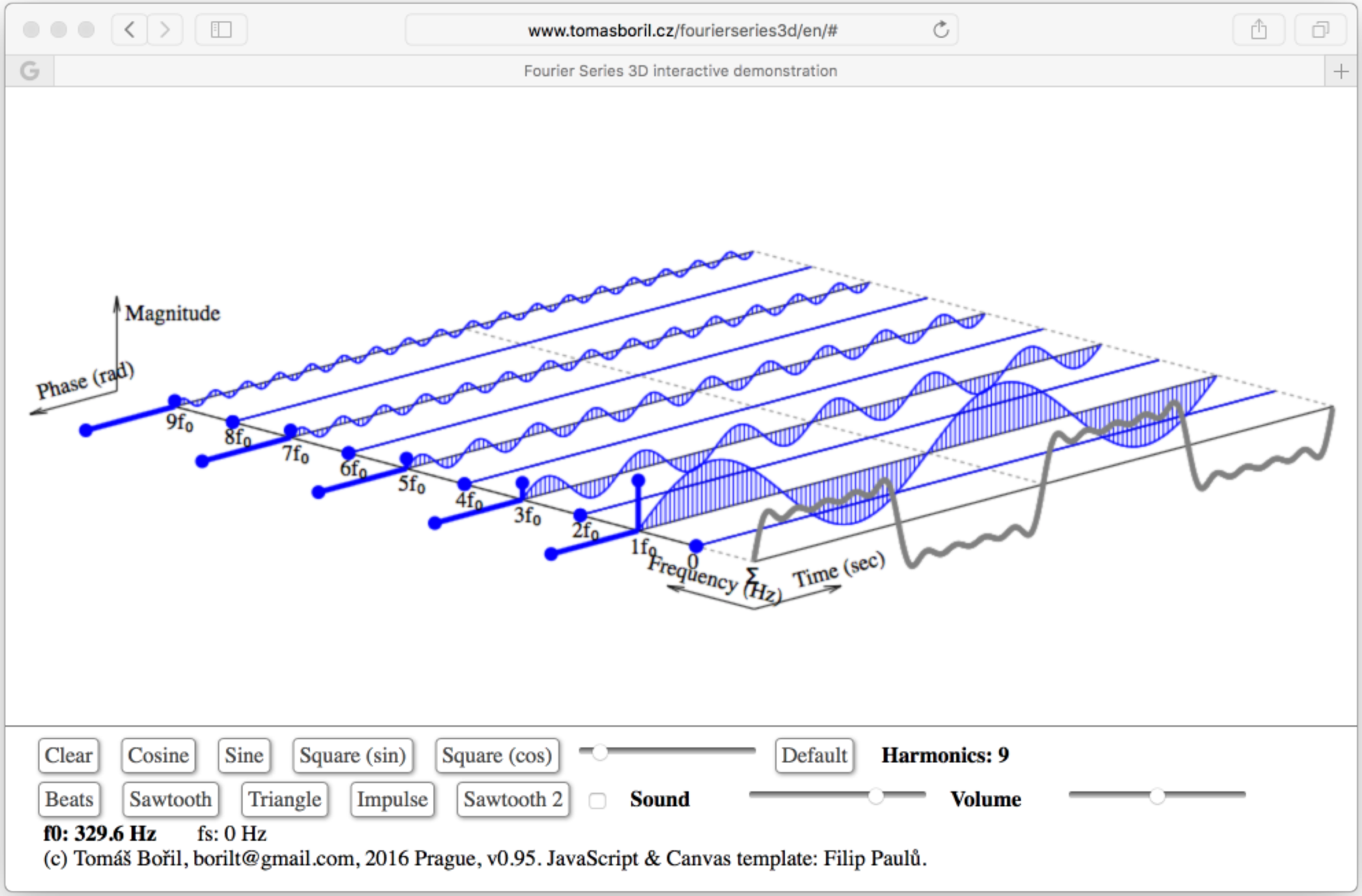
$(\theta + \phi)$ still indicates that the curve has been shifted by ϕ degrees.

Therefore…

$$
\begin{aligned}
g(x) = {} & a_1 \cos(f_1 x) + b_1 \sin(f_1 x) \\
& + a_2 \cos(f_2 x) + b_2 \sin(f_2 x) \\
& + a_3 \cos(f_3 x) + b_3 \sin(f_3 x) \\
& + \cdots
\end{aligned}
$$

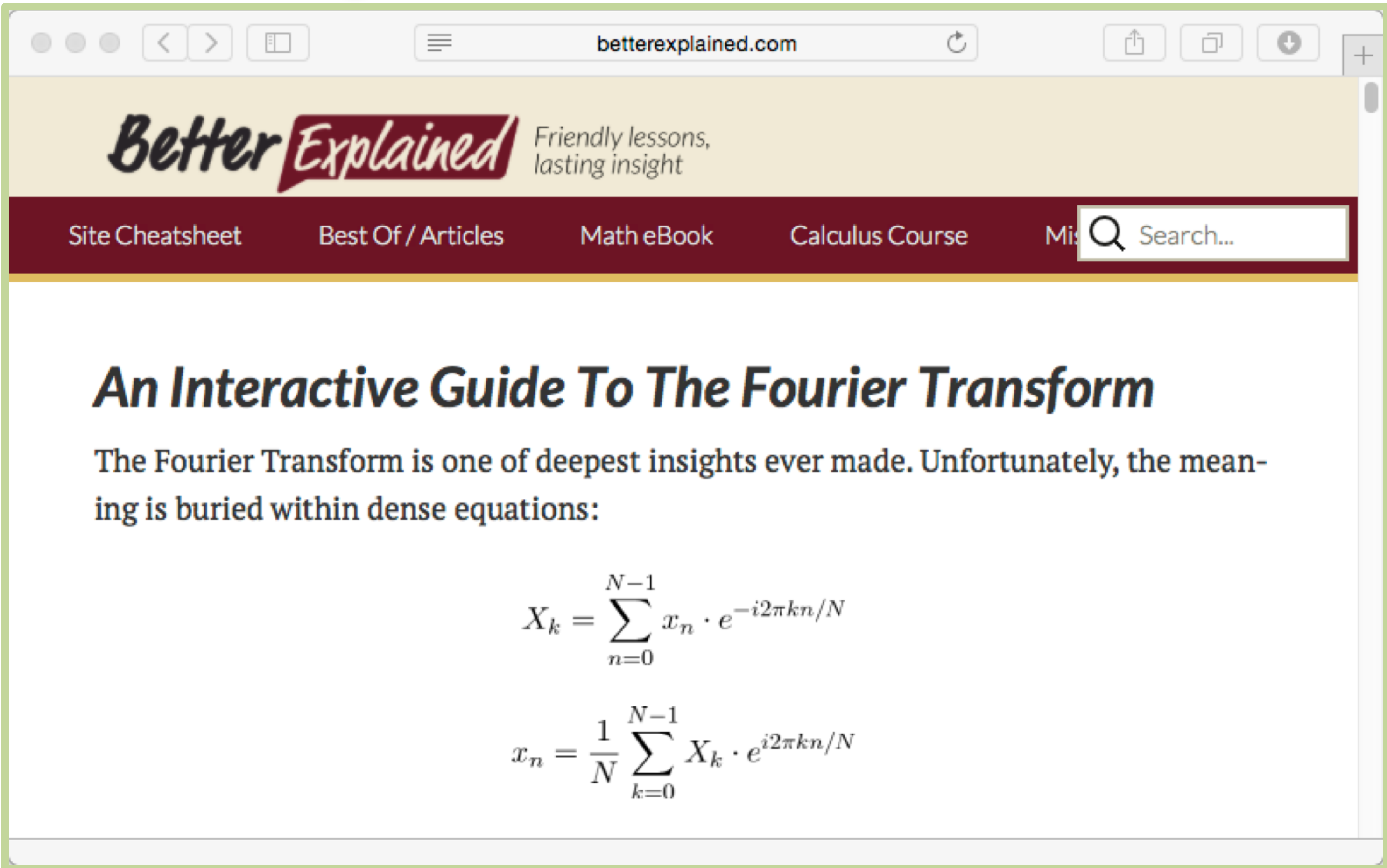
- Remaining problems
  - Now has twice as many terms (2 per frequency)
  - Must specify amplitude and frequency for each

Visualization – Nice Introduction



http://www.tomasboril.cz/fourierseries3d/en/#

Another great resource …



An Interactive Guide To The Fourier Transform

The Fourier Transform is one of deepest insights ever made. Unfortunately, the meaning is buried within dense equations:

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-i2\pi kn/N}$$

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k \cdot e^{i2\pi kn/N}$$

Colorado State University

… I forgot to mention …

en.wikipedia.org

Create account Log in

Article Talk Read Edit View history Search

WIKIPEDIA
The Free Encyclopedia

- Main page
- Contents
- Featured content
- Current events
- Random article
- Donate to Wikipedia
- Wikimedia Shop

Interaction

- Help
- About Wikipedia
- Community portal
- Recent changes
- Contact page

Euler's formula

From Wikipedia, the free encyclopedia

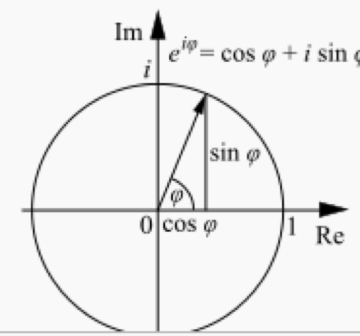
This article is about Euler's formula in complex analysis. For Euler's formula in algebraic topology and polyhedral combinatorics see Euler characteristic.

Euler's formula, named after Leonhard Euler, is a mathematical formula in complex analysis that establishes the fundamental relationship between the trigonometric functions and the complex exponential function. Euler's formula states that, for any real number *x*,

$$e^{ix} = \cos x + i \sin x$$

where *e* is the base of the natural logarithm, *i* is the imaginary unit, and cos and sin are the trigonometric functions cosine and sine respectively, with the argument *x* given in radians. This complex exponential function is sometimes denoted **cis**(*x*) ("**c**osine plus ***i*** **s**ine"). The formula is still

Part of a series of articles on the
mathematical constant *e*



Now, Per Frequency

- For any given frequency f_1 , we must calculate the amplitude of the cosine and sine functions
- g(x) is infinite (x can be anything), so we have to fit the cosines and sines to all the data:

$$a_1 = \int_{-\infty}^{+\infty} f(x)\cos(f_1 x)\,dx \qquad b_1 = \int_{-\infty}^{+\infty} f(x)\sin(f_1 x)\,dx$$

Representing Frequencies

- Our function g(x) is infinitely repeating
- Assume, WLOG, the unit of repetition is 1
  - So g(0+x) = g(1+x) = g(2+x)…
- Only cosines & sines with periods that are integers make sense
  - Otherwise it wouldn't repeat
- The cosine whose period is 1 is written as

$$\cos(2\pi x)$$

Frequencies (cont).

- In general the frequency that repeats u times over the interval is written as

$$\cos\left(2\pi u x\right)$$

- Where u is any integer

So, to rewrite g(x)

- For every integer frequency u = 1, 2, 3, …

$$\begin{aligned} g(x) = {} & a_1 \cos(2\pi x) + b_1 \sin(2\pi x) \\ & + a_2 \cos(2\pi 2x) + b_2 \sin(2\pi 2x) \\ & + a_3 \cos(2\pi 3x) + b_3 \sin(2\pi 3x) \\ & + a_4 \cos(2\pi 4x) + b_4 \sin(2\pi 4x) \\ & + \ldots \end{aligned}$$

$$a_u = \int_{-\infty}^{\infty} g(x) \cos(2\pi u x)\, dx \qquad b_u = \int_{-\infty}^{\infty} g(x) \sin(2\pi u x)\, dx$$

Think about bounds

Fourier Transform

- Formally, the Fourier transform in 1D is:

$$F(u) = \int_{-\infty}^{+\infty} f(x)\left[\cos 2\pi ux - i\sin 2\pi ux\right]dx$$

Where:

u is an integer in the range from 0 to ∞

–i is used to create a 2D vector space

$F(u) = a_u + ib_u$

The DC component

- What happens when u = 0?
  - cos(0) = 1, sin(0) = 0
- So

$$F(u=0) = \int_{-\infty}^{+\infty} f(x)\left[\cos 2\pi ux - i\sin 2\pi ux\right]dx$$

$$= \int_{-\infty}^{+\infty} f(x)dx$$

  - This is the average value (or "DC component") of the function. For images, it is largely a function of lighting and camera gain.

Colorado State University

Inverse Fourier Transform

- What if I have F(u) for all u, and I want to recreate the original function g(x)?

- Well, sum it up for every u:

$$f(x) = \int_{-\infty}^{+\infty} F(u)\left[\cos(2\pi ux) + i\sin(2\pi ux)\right]du$$

Discrete Fourier Transform

- Problem: an image is not an analogue signal that we can integrate.
- Therefore for $0 \le x < N$ and $0 \le u < N/2$:

$$F(u) = \sum_{x=0}^{N-1} f(x)\left[\cos\left(\frac{2\pi ux}{N}\right) - i\sin\left(\frac{2\pi ux}{N}\right)\right]$$

And the discrete inverse transform is:

$$f(x) = \frac{1}{N}\sum_{x=0}^{N-1} F(u)\left[\cos\left(\frac{2\pi ux}{N}\right) + i\sin\left(\frac{2\pi ux}{N}\right)\right]$$

Discrete vs. Continuous

- Summation replaces integration
- Division by N (the number of discrete samples) makes the unit of repetition 1.
- For any signal (continuous or discrete)
  - G(x) is called the spatial domain
  - F(u) is called the frequency domain

Colorado State University

Spatial vs Frequency

- Spatial domain representation size?
  - Given N samples, it is size N
- Frequency domain representation size?
  - A total of N/2 frequencies
  - A complex number (2 values) per frequency
- The DFT is invertible, so the two representations are equivalent:
  - Exact same information and same size
  - The DFT is O(n log n)