Optimal Reliability Allocation

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Reliability Allocation Problem

- Allocation the reliability values to subsystems
  - to minimize the total cost
  - while achieving the reliability target.

- Widely applicable
  - Software systems
  - Electrical systems
  - Mechanical systems

- Implementation choices
  - Discrete
  - Continuous
Reliability Allocation in Software

- A software system consists of many functional modules
  - Some reused, probably with lower defect densities
  - Some are new, with higher defect densities
  - Some are invoked more often

- To increase reliability
  - Additional testing
  - Replicated using n-version programming?

- What is the best strategy?
Optimal Reliability Allocation

- **System composed of subsystems:**
  - Subsystem cost a function of reliability
  - System reliability depends on subsystems
  - Failure rate as a reliability measure

- **Commons systems:** series and parallel

- **Software system reliability**
  - Fractional execution time
  - Lagrange multiplier: closed form optimal solution
  - Parameter dependence: size, defect density

- **Apportionment & general approach**
Problem Formulation

- System S has subsystems $S_{si}$, $i = 1, \ldots, n$.
- Each subsystem $SS_{Si}$ has a specific functionality.
- Several choices with same functionality, but differently reliability levels.

$$C_i = f_i(R_i)$$

- Minimize system cost

$$C_s = \sum_{i=1}^{n} C_i = \sum_{i=1}^{n} f_i(R_i)$$

- Subject to

  target system reliability $R_{ST}$
  
  $$\leq$$ achieved reliability $R_s$$
Cost minimization problem

Minimize $C_s = \sum_{i=1}^{n} C_i = \sum_{i=1}^{n} f_i(R_i)$

Subject to $R_{ST} \leq R_s$

For a series system $R_s = \prod_{i=1}^{n} R_i$

thus $R_{ST} \leq \prod_{i=1}^{n} R_i$
Subsystem implementation choices

- Subsystem can be made more reliable by extending a continuous attribute
  - diameter of a column in building
  - time spent for software testing.
- Different vendors implementations of SSi at different costs.
- Multiple copies of SSi to achieve higher reliability.
  - double wheels of a truck
- Number of copies is constrained between one and a practical number because of implementation issues.
The Cost function

Cost function $f_i$ should satisfy these three conditions:

- $f_i$ is a positive function
- $f_i$ is non-decreasing, thus higher reliability will come at a higher cost.
- $f_i$ increases at a higher rate for higher values of $R_i$

Mettas A, Reliability allocation and optimization for complex systems. Pro Ann Reliability and Maintainability Symp, January 2000, 216-221
In terms of failure rate

- Taking log of both sides, and since \( R_i(t) = e^{-\lambda_i t} \)

\[
\ln( R_{ST} ) \leq \sum_{i=1}^{n} \ln( R_i ) \quad \lambda_{ST} \geq \sum_{i=1}^{n} \lambda_i
\]

- Stating cost as a function of failure rate

\[
C_s = \sum_{i=1}^{n} C_i = \sum_{i=1}^{n} f_i(\lambda_i)
\]
In terms of failure rate: SRGM

- exponential software reliability growth model

\[ \lambda_i(d) = \lambda_{0i} \exp(-\beta_i d) \]

- \( \lambda_{0i} \) depends on initial defect density
- \( \beta_i \) depends inversely on program size

- Restating it as Cost function

\[ d(\lambda_i) = \frac{1}{\beta_i} \ln \left( \frac{\lambda_{0i}}{\lambda_i} \right) \]

Assumes constant development cost, thus neglected
Series and Parallel Systems: linearization

- Constraint Linearization simplifies the calculations.
- Series system
  \[ \ln(R_{ST}) \leq \sum_{i=1}^{n} \ln(R_i) \]
- Parallel system: log of unreliaibites
  \[ R_{ST} \leq 1 - \prod_{i=1}^{n} (1 - R_i) \quad \ln(1 - R_{ST}) \geq \sum_{i=1}^{n} (\ln(1 - R_i)) \]

- Elegbede: If cost function satisfies 3 properties given above, the cost is optimal if all parallel components have the same cost.
Reliability Allocation for Software Systems

- A block $i$ is under execution for a fraction $x_i$ of the time where $\sum x_i = 1$
- Reliability allocation problem

\[
\text{Minimize } C = \sum_{i=1}^{n} \frac{1}{\beta_i} \ln \left( \frac{\lambda_{0i}}{\lambda_i} \right)
\]

Subject to $\lambda_{ST} \geq \sum_{i=1}^{n} x_i \lambda_i$
Solution using Lagrange multiplier

- solutions for the optimal failure rates

$$\lambda_1 = \frac{\lambda_{ST}}{\sum_{i=1}^{n} \frac{\beta_i}{\beta_1}}$$

$$\lambda_2 = \frac{\beta_1 x_1}{\beta_2 x_2} \lambda_1 \quad \ldots \quad \lambda_n = \frac{\beta_1 x_1}{\beta_n x_n} \lambda_1$$

- optimal values of test times $d_1$ and $d_i$, $i \neq 1$

$$d_1 = \frac{1}{\beta_1} \ln \left( \frac{\lambda_{10} x_1 \sum_{i=1}^{n} \frac{\beta_1}{\beta_i}}{\lambda_{ST}} \right)$$

$$d_i = \frac{1}{\beta_i} \ln \left( \frac{\lambda_{i0} \beta_i x_i}{\lambda_1 \beta_1 x_1} \right)$$
Observations: Software reliability allocation

- A reused subsystem have a higher reliability because of past testing causing $\lambda_i \geq \lambda_{i0}$ and hence negative $d_i$.
  - Solution: apply allocation problem only to modules with positive $d_i$.

- If $x_i$ is proportional to the subsystem code size, then optimal values of the post-test failure rates $\lambda_1, \ldots, \lambda_n$ are equal.
Ex: Optimal: Software with 5 blocks

<table>
<thead>
<tr>
<th>Block</th>
<th>B₁</th>
<th>B₂</th>
<th>B₃</th>
<th>B₄</th>
<th>B₅</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size KSLOC</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>Ini Defect density</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>15</td>
<td>20</td>
</tr>
<tr>
<td>$\beta_i$</td>
<td>$4.59 \times 10^{-3}$</td>
<td>$2.30 \times 10^{-3}$</td>
<td>$1.53 \times 10^{-3}$</td>
<td>$4.59 \times 10^{-4}$</td>
<td>$2.30 \times 10^{-4}$</td>
</tr>
<tr>
<td>$\lambda_{i0}$</td>
<td>0.046</td>
<td>0.046</td>
<td>0.046</td>
<td>0.069</td>
<td>0.092</td>
</tr>
<tr>
<td>$x_i$</td>
<td>0.028</td>
<td>0.056</td>
<td>0.083</td>
<td>0.278</td>
<td>0.556</td>
</tr>
<tr>
<td>Optimal $\lambda_i$</td>
<td>0.04</td>
<td>0.04</td>
<td>0.04</td>
<td>0.04</td>
<td>0.04</td>
</tr>
<tr>
<td>Optimal $d_i$</td>
<td>30.1</td>
<td>60.1</td>
<td>90.2</td>
<td>1184</td>
<td>3620</td>
</tr>
</tbody>
</table>

- Optimal when all modules have the same failure rate!
Ex: Equal testing

<table>
<thead>
<tr>
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</tr>
<tr>
<td>( \beta_i )</td>
<td>4.59\times10^{-3}</td>
<td>2.30\times10^{-3}</td>
<td>1.53\times10^{-3}</td>
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</tr>
<tr>
<td>( \lambda_i )</td>
<td>0.146</td>
<td>0.003</td>
<td>0.01</td>
<td>0.08</td>
<td>0.15</td>
</tr>
<tr>
<td>Equal ( d_i )</td>
<td>1109.4</td>
<td>1109.4</td>
<td>1109.4</td>
<td>1109.4</td>
<td>1109.4</td>
</tr>
</tbody>
</table>

If Total test time is equally distributed for all 5 blocks, system will have significantly higher failure rate of **0.055** per unit time.
Ex: Testing only B5

If Total test time is allowed only for block B5, system will have higher failure rate of **0.043** per unit time.

<table>
<thead>
<tr>
<th>Block</th>
<th>B1</th>
<th>B2</th>
<th>B3</th>
<th>B4</th>
<th>B5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size KSLOC</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>Ini Defect density</td>
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<td>10</td>
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<td>0.046</td>
<td>0.046</td>
<td>0.069</td>
<td>0.092</td>
</tr>
<tr>
<td>$x_i$</td>
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<td>0.056</td>
<td>0.083</td>
<td>0.278</td>
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<td>0.003</td>
<td>0.01</td>
<td>0.08</td>
<td>0.15</td>
</tr>
<tr>
<td>Equal $d_i$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>5547</td>
</tr>
</tbody>
</table>
Illustration using excel

- See Excel sheet relallocationexamples.xls
- Try changing entries.
Common Apportionment rules

- **Equal reliability apportionment:**
  - At end they all individually have failure rate equal to target failure rate for the system

- **Complexity based apportionment**
  - Test time apportioned in proportion to the software size

- **Impact based apportionment:**
  - A component executed more frequently, or more critical, should be assigned more resources
Reliability Allocation for Complex Systems

- An iterative approach
  - Design the system using functional subsystems.
  - Perform an initial apportionment of cost or reliability attributes based on suitable apportionment rules or preliminary computation.
  - Predict system reliability.
  - Is reallocation feasible and will enhance the objective function. If so, perform reallocation.
  - Repeat until optimality is achieved.
  - Does this meets objectives? If not, return to step 1 and revising the design at a higher level..
Conclusions

- Reliability allocation: consider how cost varies with reliability.

- Software testing:
  - cost $\propto \log(1/\text{failure rate})$
  - $\beta_1 \propto \text{size}$

- Reliability allocation in systems with replicated subsystems can encounter correlated failures and thus would need a more careful modeling.