Reliability Analysis: Outline

Reliability measures:
- **Reliability, availability, Transaction Reliability,**
- **MTTF and R(t), MTBF**

Basic Cases
- **Single unit with permanent failure, failure rate**
- **Single unit with temporary failures**

Combinatorial Reliability: Block Diagrams
- **Serial, parallel. K-out-of-n systems**
- **Imperfect coverage**

Redundancy
- **TMR, spares**
- **Generalized**

Reliability Analysis

- **Permanent faults**
  - **The unit will eventually fail. Thus reliability "decays".**
- **Temporary faults**
  - **Faults come and go. Often Steady state characterization is possible.**
  - **Permanent faults subject to repair are modeled as temporary faults.**
- **Design faults**
  - **Reliability growth occurs during testing & debugging. We will study this under "Software Reliability" later.**

Why Mathematical Analysis?

- **You can determine reliability by constructing a large number of copies of the target system, and collecting failure data. However, that would be infeasible except for special cases.**
- **Thus we need to be able to determine the reliability before a system is built, by using the information we have about the components and the proposed architecture.**

Basic Reliability Measures

- **Reliability:** durational (default)
  R(t)=P{correct operation in duration (0,t)}
  - This is the default definition of reliability.
- **Availability:** instantaneous
  A(t)= P{correct operation at instant t)}
  - Applied in presence of temporary failures
  - A steady-state value is the expected value over a range of time.
- **Transaction Reliability:** single transaction
  R_t=P{a transaction is performed correctly}
- The term "Reliability" is sometimes used with a non-standard meaning.

Mean time to …

- **Mean Time to Failure (MTTF): expected time the unit will work without a failure.**
- **Mean time between failures (MTBF): expected time between two successive failures.**
  - **Applicable when faults are temporary.**
  - **The time between two successive failures includes repair time and then the time to next failure.**
  - **Approximately equal to**
- **Mean time to repair (MTTR): expected time during which the unit is non-operational.**

Mean Time to Failure (MTTF)

- There is a very useful general relation between MTTF and R(t). Here T is time to failure, which is a random variable.

$$MTTF = E(T) = \int_0^\infty t\, f(t)dt$$

$$= -\int_0^\infty t \frac{dR(t)}{dt} dt$$

$$= [-tR(t)]_0^\infty + \int_0^\infty R(t)dt$$

$$Thus\ \boxed{MTTF = \int_0^\infty R(t)dt}$$

Worth Remembering!

Note:

$$R(t) = 1 - P\{failure\ in\ (0,t)\}$$
$$= 1 - P\{0 \le T \le t\}$$
$$= 1 - F(t)$$
$$\therefore \frac{dF(t)}{dt} = -\frac{dR(t)}{dt}$$
$$or\ f(t) = -\frac{dR(t)}{dt}$$

Note:

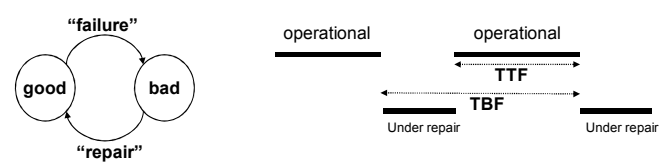
$xe^{-x} \to 0\ as\ x \to 0$

and R(t) is generally of the form e^{-at}

Thus $tR(t) \to 0\ as\ t \to \infty$.

Failures with Repair

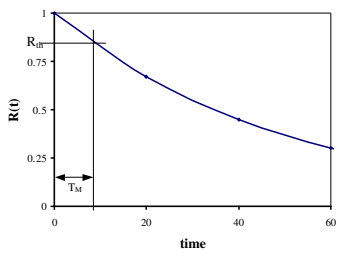
- **Time between failures: time to repair + time to next failure**



- **MTBF = MTTF + MTTR**
  - MTBF, MTTF are same same when MTTR ≈0
- **Steady state availability = MTTF / (MTTF+MTTR)**

Mission Time (High-Reliability Systems)

- Reliability throughout the missio must remain above a threshold reliability R_{th}.
- Mission time T_M: defined as the duration in which $R(t) \geq R_{th}$.
- R_{th} may be chosen to be perhaps 0.95.
- Mission time is a strict measure, used only for very high reliability missions.



Two Basic cases

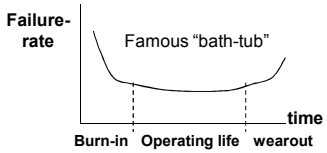
- We next consider two very important basic cases that serve as the basis for time-dependent analysis.

1. Single unit subject to permanent failure
   - We will assume a constant failure rate to evaluate reliability and MTTF.
2. Single unit with temporary failures
   - System has two states Good and Bad, and transitions among them are defined by transition rates.

- Both of these are example of Markov processes.

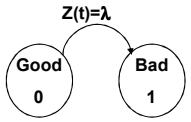
Constant Failure Rate Assumption

- We will always assume a constant failure rate.
  - It keeps analysis simple.
  - During operating life, the failure rate is approximately constant.
- The Bath-Tub curve:
  - In the beginning the failure rate is high because the weaker devices fail due to "infant mortality". Near the end the failure rate is again high due to "aging" or wear-out of devices.



Basic Cases: Single Unit with Permanent Failure

- Failure rate is the probability of failure/unit time
- Assumption: constant failure-rate λ



The state transition diagram & the differential equation represent What we call Markov Modeling.

$$\frac{dp_0(t)}{dt} = -\lambda\ p_0(t)$$ since the rate of leaving state 0 depends on probability of being in state 0

$p_0(0) = 1$ initial condition

Single Unit with Permanent Failure (2)

$$\frac{dp_0(t)}{dt} = -\lambda\, p_0(t)$$

$$p_0(0) = 1$$

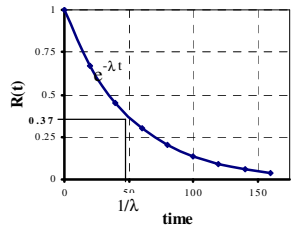
$$Solution: p_0(t) = e^{-\lambda t}$$

$$Since\ R(t) = p_0(t)$$

$$\boxed{R(t) = e^{-\lambda t}}$$

"The Exponential reliability law"

$$At \quad t = \frac{1}{\lambda},\ R(t) = e^{-1} = 0.368$$



Single Unit: Permanent Failure (3)

$$R(t) = e^{-\lambda t}$$

A(t) is same as R(t) in this case.

$$MTTF = \int_0^\infty R(t)dt = \int_0^\infty e^{-\lambda t}dt$$

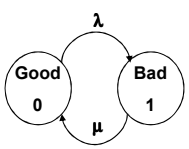
$$= [-\frac{e^{-\lambda t}}{\lambda}]_0^\infty$$

$$= \frac{1}{\lambda}$$

- **Ex 1:** a unit has MTTF =30,000 hrs. Find failure rate. λ=1/30,000=3.3x10^{-5}/hr
- **Ex 2:** Compute mission time T_M if R_{th} =0.95. $e^{-\lambda T_M}$ =0.95 T_M= - ln(0.95)/ λ ≈0.051/ λ
- **Ex 3:** Assume λ=3.33x10$^{-5,}$ and R_{th} =0.95 find $T_{M.}$ Ans: T_M = 1538.8 hrs (compare with MTTF =30,000)

Single Unit: Temporary Failures(1)

- **Temporary: intermittent, transient, permanent with repair**



$$\frac{dp_0(t)}{dt} = -\lambda\, p_0(t) + \mu\, p_1(t)$$

$$\frac{dp_1(t)}{dt} = +\lambda\, p_0(t) - \mu\, p_1(t)$$

Note state diagram & Differential equations for Markov modeling

can be solved by laplace transform etc.

$$p_0(t) = p_0(0)e^{-(\lambda+\mu)t} + \frac{\mu}{\lambda+\mu}(1 - e^{-(\lambda+\mu)t})$$

Y. K. Malaiya, S. Y. H. Su: Reliability Measure of Hardware Redundancy Fault-Tolerant Digital Systems with Intermittent Faults. IEEE Trans. Computers 30(8): 600-604 (1981)

Similarly we can get an expression for p_1(t), however it is not needed since $p_1(t) = 1 - p_0(t)$.

Single Unit: Temporary Failures(2)

- $p_0(t) = p_0(0)e^{-(\lambda+\mu)t} + \frac{\mu}{\lambda+\mu}(1 - e^{-(\lambda+\mu)t})$
- Availability A(t) = $p_0(t)$

Thus $A(t) = p_0(0)e^{-(\lambda+\mu)t} + \frac{\mu}{\lambda+\mu}(1 - e^{-(\lambda+\mu)t})$

- Note that steady – state probabilities exist :

$$t \to \infty,\ p_0(t) = \frac{\mu}{\lambda+\mu} \qquad p_1(t) = \frac{\lambda}{\lambda+\mu}$$

- Steady - state availability is $\frac{\mu}{\lambda+\mu}$

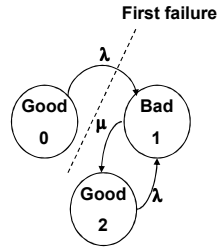
Single Unit: Temporary Failures(3)

- Reliability (durational)

$$R(t) = P\{\text{no failures in } (0,t)\}$$
$$= P\{in\ Good\,0\ at\ t\}$$
$$= e^{-\lambda t}$$

same as permanent failure

- Thus $\text{MTTF} = \dfrac{1}{\lambda}$
- Mission time : also same



Note that when we say **no failures in (0,t)**, even a brief failure is a failure. Thus R(t) may be too strict a measure when brief failures may be acceptable.

Combinatorial Reliability

This is a part of classic reliability theory.

Objective is: Given a

- systems structure in terms of its units
- reliability attributes of the units
- some simplifying assumptions

- We need to **evaluate the overall reliability** measure.

There are **two extreme cases** we will examine first:

- Series configuration
- Parallel configuration
- Other cases involve combinations and other configurations.

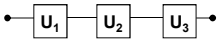
- Note that conceptual modeling is applicable to R(t), A(t), $R_t(t)$. A system is either good or bad.

Series configuration

Series configuration: all units are essential. System fails if one of them fails .

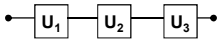
- **Assumption:** statistically independent failures in units.

$$R_S = P\{U_1\ good \cap U_2\ good \cap U_3\ good\}$$
$$= P\{U_1\ g\}P\{U_2\ g\}P\{U_3\ g\}$$
$$= R_1 R_2 R_3$$



In general $R_S = \prod_{i=1}^{n} R_i$

Series configuration



If $R_i(t) = e^{-\lambda_i t}$

then $R_s(t) = \Pi e^{-\lambda_i t} = e^{-[\lambda_1 + \lambda_2 + \cdots + \lambda_n]t}$

i.e. system failure rate is the sum of individual failure rates :

$\lambda_S = \lambda_1 + \lambda_2 + \cdots + \lambda_n$

The reliability block diagrams like this are only conceptual, not physical.

This gives us a nice way to estimate the overall failure rate, when all the individual units are essential. This is the basis of the approach used in the popular "Military Handbook" MIL-HDBK-217 approach for estimating the failure rates for different systems.

The failure rates of individual units are estimated using empirical formulas. For example the failure rate of a VLSI chip is related to its complexity etc.

"A chain is as strong as it's weakest link"

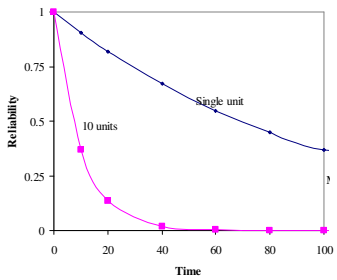
Do you agree?

Let us see for a 4-unit series system

- Assume $R_1=R_2=R_3=0.95$, $R_4=0.75$
  - $R_S=0.643$
- Thus a chain is slightly weaker than its weakest link!

The plot gives reliability of a 10-unit system vs a single system. Each of the 10 units are identical.

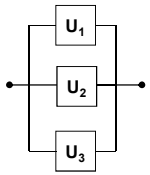
- More units, less reliability.



Combinatorial: Series

Combinatorial: Parallel

- **Parallel configuration:** System is good when least one of the several replicated units is good. A parallel configuration represents an *ideal* redundant system, ignoring any overhead.



$$R_s = 1 - P\{all\ units\ bad\}$$
$$= 1 - P\{U_1\ bad \cap U_2\ bad \cap U_3\ bad\}$$
$$= 1 - P\{U_1\ b.\}P\{U_2\ b.\}P\{U_3\ b.\}$$
$$= 1 - (1-R_1)(1-R_2)(1-R_3)$$

$$\text{In general } R_s = 1 - \prod_{i=1}^{n}(1-R_i)$$

$$i.e.\quad \overline{R}_s = \prod_{i=1}^{n}\overline{R}_i$$

Where $\overline{R}$ represents 1-R, i.e. "*unreliability*"

Combinatorial: Parallel

Parallel Configuration: Example

Problem: Need system reliability $R_s = 1-\in$
How many parallel units are needed
if $R_1 = R_2 = \cdots = R_m$, $R_m < R_s$?

Sometimes it is more convenient to talk in terms of "*unreliability*"

Solution: $1 - R_s = (1-R_m)^x$

$$\in = (1-R_m)^x$$

$$x = \frac{\ln \in}{\ln(1-R_m)}$$

Assume $R_s = 0.9999$ ($\in = 0.0001$),
$R_m = 0.9$
gives $x = 4$.

Remember, we're consider an *ideal* system

Combinatorial: Parallel