CS 301 - Lecture 5
Regular Grammars, Regular Languages, and Properties of Regular Languages

Fall 2008

Review

- Languages and Grammars
  - Alphabets, strings, languages
- Regular Languages
  - Deterministic Finite Automata
  - Nondeterministic Finite Automata
  - Equivalence of NFA and DFA
  - Regular Expressions
- Today:
  - Regular Grammars and Regular Languages
  - Properties of Regular Languages

Grammars

- Grammars express languages
- Example: the English language

$$\langle sentence \rangle \rightarrow \langle noun\_phrase \rangle \ \langle predicate \rangle$$

$$\langle noun\_phrase \rangle \rightarrow \langle article \rangle \ \langle noun \rangle$$

$$\langle predicate \rangle \rightarrow \langle verb \rangle$$

Grammar Notation

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Production Rules

$$\langle noun \rangle \rightarrow cat$$

$$\langle noun \rangle \rightarrow dog$$

Variable

Terminal

Some Terminal Rules

$$\langle article \rangle \rightarrow a$$
$$\langle article \rangle \rightarrow the$$

$$\langle noun \rangle \rightarrow cat$$
$$\langle noun \rangle \rightarrow dog$$

$$\langle verb \rangle \rightarrow runs$$
$$\langle verb \rangle \rightarrow walks$$

A Resulting Sentence

$$\begin{aligned}\langle sentence \rangle &\Rightarrow \langle noun\_phrase \rangle \ \langle predicate \rangle \\ &\Rightarrow \langle noun\_phrase \rangle \ \langle verb \rangle \\ &\Rightarrow \langle article \rangle \ \langle noun \rangle \ \langle verb \rangle \\ &\Rightarrow the \ \langle noun \rangle \ \langle verb \rangle \\ &\Rightarrow the \ \ dog \ \langle verb \rangle \\ &\Rightarrow the \ \ dog \ \ walks\end{aligned}$$

The Resulting Language

L = { "a cat runs",
"a cat walks",
"the cat runs",
"the cat walks",
"a dog runs",
"a dog walks",
"the dog runs",
"the dog walks" }

Definition of a Grammar

$$G = (V, T, S, P)$$

V: Set of variables

T: Set of terminal symbols

S: Start variable

P: Set of Production rules

A Simple Grammar

- Grammar: $S \to aSb$
  $S \to \lambda$

- Derivation of sentence : ab

$$S \Rightarrow aSb \Rightarrow ab$$

$S \to aSb$ $\quad$ $S \to \lambda$

Example Grammar Notation

$$S \to aSb$$
$$S \to \lambda$$

$$G = (V, T, S, P)$$

$V = \{S\}$ $\quad$ $T = \{a, b\}$

$P = \{S \to aSb,\ S \to \lambda\}$

Deriving Strings in the Grammar

- Grammar:

$$S \to aSb$$
$$S \to \lambda$$

- Derivation of sentence $aabb$:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aabb$$

$S \to aSb$ $\quad$ $S \to \lambda$

Sentential Form

- A sentence that contains variables and terminals

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb$$

Sentential Forms $\quad$ sentence

General Notation for Derivations

- In general we write: $w_1 \overset{*}{\Rightarrow} w_n$
- If: $w_1 \Rightarrow w_2 \Rightarrow w_3 \Rightarrow \cdots \Rightarrow w_n$
- It is always the case that: $w \overset{*}{\Rightarrow} w$

Why Notation Is Useful

- We can now write:

$$S \overset{*}{\Rightarrow} aaabbb$$

- Instead of:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb$$

Language of a Grammar

Grammar can produce some set of strings

Set of strings over an alphabet is a language

Language of a grammar is all strings produced by the grammar

$$L(G) = \{w: \quad S \overset{*}{\Rightarrow} w\}$$

String of terminals

Example Language

$$S \to aSb$$

$$S \to \lambda$$

Consider the set of all strings that can derived from this grammar…..

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb$$

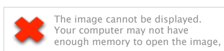
$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb$$
$$\Rightarrow aaaaSbbbb \Rightarrow aaaabbbb$$

What language is being described?

The Resulting Language

$$S \to aSb$$
$$S \to \lambda$$

Always add on a and b on each side resulting in:
a's at the left
b's at the right
equal number of a's and b's



Linear Grammars

- Grammars with at most one variable at the right side of a production
- Examples: $S \to aSb$
  $S \to \lambda$

A Non-Linear Grammar

Grammar G :

$$S \to SS$$
$$S \to \lambda$$
$$S \to aSb$$
$$S \to bSa$$

$$L(G) = \{w : n_a(w) = n_b(w)\}$$

Number of $a's$ in string w

Another Linear Grammar

Grammar G :

$$S \to A$$
$$A \to aB \mid \lambda$$
$$B \to Ab$$

$$L(G) = \{a^n b^n : n \geq 0\}$$

Right-Linear Grammars

- All productions have form:

$$A \to xB$$

or

$$A \to x$$

string of terminals

- Example:

$$S \to abS$$
$$S \to a$$

Left-Linear Grammars

- All productions have form:

$$A \to Bx$$

or

$$A \to x$$

string of terminals

- Example:

$$S \to Aab$$
$$A \to Aab \mid B$$
$$B \to a$$

Regular Grammars

Regular Grammars

- A regular grammar is any right-linear or left-linear grammar

- Examples:

$$S \to abS$$
$$S \to a$$

$$S \to Aab$$
$$A \to Aab \mid B$$
$$B \to a$$

What languages are generated by these grammars?

Languages and Grammars

$$S \to abS$$
$$S \to a$$

$$S \to Aab$$
$$A \to Aab \mid B$$
$$B \to a$$

$L(G_1) = (ab)^* a$ $\quad$ $L(G_2) = aab(ab)^*$

Note both these languages are regular
- we have regular expressions for these languages (above)
- we can convert a regular expression into an NFA (how?)
- we can convert an NFA into a DFA (how?)
- we can convert a DFA into a regular expression (how?)

Do regular grammars also describe regular languages??

Regular Grammars Generate Regular Languages

Theorem

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{Generated by} \\ \text{Regular Grammars} \end{array} \right\} = \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

Theorem - Part 1

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{Generated by} \\ \text{Regular Grammars} \end{array} \right\} \subseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

Any regular grammar generates a regular language

Proof – Part 1

$$\left\{\begin{array}{l}\text{Languages}\\ \text{Generated by}\\ \text{Regular Grammars}\end{array}\right\} \subseteq \left\{\begin{array}{l}\text{Regular}\\ \text{Languages}\end{array}\right\}$$

The language $L(G)$ generated by any regular grammar G is regular

The case of Right-Linear Grammars

- Let G be a right-linear grammar
- We will prove: $L(G)$ is regular
- **Proof idea:** We will construct NFA using the grammar transitions

Example

Given right linear grammar:

$$V_0 \rightarrow aV_1$$
$$V_1 \rightarrow abV_0 | b$$

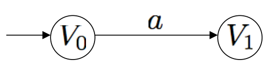
Step 1: Create States for Each Variable

- Construct NFA M such that every state is a grammar variable:

$$V_0 \rightarrow aV_1$$
$$V_1 \rightarrow abV_0 | b$$

Step 2.1:
Edges for Productions

- Productions of the form $V_i \to aV_j$ result in $\delta(V_i, a) = V_j$

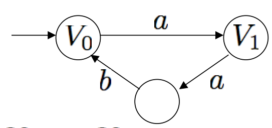


$V_0 \to aV_1$

$V_1 \to abV_0|b$

Step 2.2:
Edges for Productions

- Productions of the form $V_i \to wV_j$ are only slightly harder…. Create row of states that derive w and end in V_j

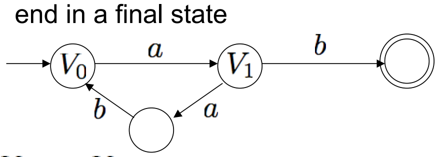


$V_0 \to aV_1$

$V_1 \to abV_0|b$

Step 2.3:
Edges for Productions

- Productions of the form $V_i \to w$ Create row of states that derive w and end in a final state



$V_0 \to aV_1$

$V_1 \to abV_0|b$

In General

- Given any right-linear grammar, the previous procedure produces an NFA
  - We sketched a proof by construction
  - Result is both a proof and an algorithm
  - *Why doesn't this work for a non linear grammar?*
- Since we have an NFA for the language, the right-linear grammar produces a regular language

Proof - Part 2

$$\left\{\begin{array}{l}\text{Regular}\\ \text{Languages}\end{array}\right\} \subseteq \left\{\begin{array}{l}\text{Languages}\\ \text{Generated by}\\ \text{Regular Grammars}\end{array}\right\}$$

Any regular language L is generated by some regular grammar G

Any regular language L is generated by some regular grammar G

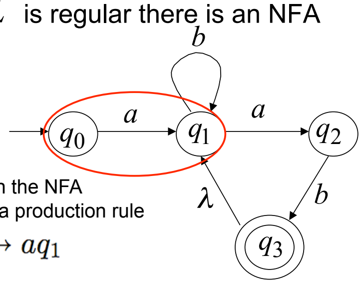
Proof idea:

Let M be the NFA with $L = L(M)$.

Construct from M a regular grammar G such that $L(M) = L(G)$

NFA to Grammar Example

- Since L is regular there is an NFA

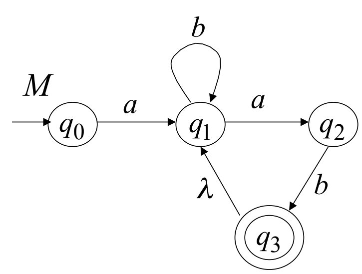


This transition in the NFA
Looks a lot like a production rule

$q_0 \rightarrow aq_1$

Step 1:
Convert Edges to Productions

$q_0 \rightarrow aq_1$
$q_1 \rightarrow bq_1$
$q_1 \rightarrow aq_2$
$q_2 \rightarrow bq_3$



Step 2: λ Edges and Final States

$q_0 \to aq_1$

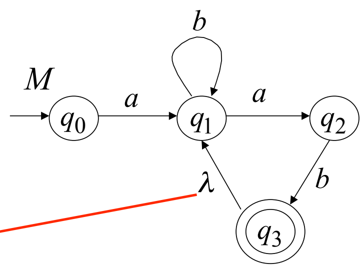
$q_1 \to bq_1$

$q_1 \to aq_2$

$q_2 \to bq_3$

$q_3 \to q_1$

$q_3 \to \lambda$



Step 2: λ Edges and Final States

$q_0 \to aq_1$

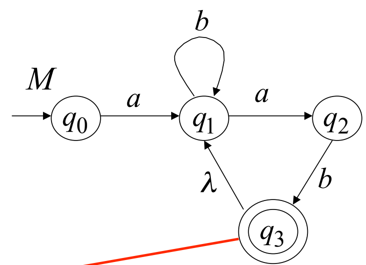
$q_1 \to bq_1$

$q_1 \to aq_2$

$q_2 \to bq_3$

$q_3 \to q_1$

$q_3 \to \lambda$



If q_i is a final state, add
$q_i \to \lambda$

In General

- Given any NFA, the previous procedure produces a right linear grammar
  - We sketched a proof by construction
  - Result is both a proof and an algorithm
- Every regular language has an NFA
  - Can convert that NFA into a right linear grammar
  - Thus every regular language has a right linear grammar
- Combined with Part 1, we have shown right linear grammars are yet another way to describe regular languages

But What About Left-Linear Grammars

- What happens if we reverse a left linear grammar as follows:

$V_i \to V_j w$ Reverses to $V_i \to w^R V_j$

$V_i \to w$ Reverses to $V_i \to w^R$

- The result is a right linear grammar.
  - If the left linear grammar produced L, then what does the resulting right linear grammar produce?

But What About Left-Linear Grammars

- The previous slide reversed the language!

$V_i \to V_j w$ Reverses to $V_i \to w^R V_j$

$V_i \to w$ Reverses to $V_i \to w^R$

- If the left linear grammar produced language L, then the resulting right linear grammar produces L^R

Claim we just proved left linear grammars produce regular languages? Why?

Left-Linear Grammars Produce Regular Languages

- Start with a Left Linear grammar that produces L want to show L is regular
- Can produce a right linear grammar that produces L^R
- All right linear grammars produce regular languages so L^R is a regular language
- The reverse of a regular language is regular so

$(L^R)^R = L$ is a regular language!

For regular languages L_1 and L_2 we will prove that:

$$\left.\begin{array}{rl} \text{Union:} & L_1 \cup L_2 \\ \text{Concatenation:} & L_1 L_2 \\ \text{Star:} & L_1 * \\ \text{Reversal:} & L_1{}^R \\ \text{Complement:} & \overline{L_1} \\ \text{Intersection:} & L_1 \cap L_2 \end{array}\right\} \text{Are regular Languages}$$

We say: Regular languages are **closed under**

Union: $L_1 \cup L_2$

Concatenation: $L_1 L_2$

Star: $L_1 *$

Reversal: $L_1{}^R$

Complement: $\overline{L_1}$

Intersection: $L_1 \cap L_2$

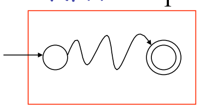
Regular language L_1

Regular language L_2

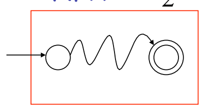
$L(M_1) = L_1$

$L(M_2) = L_2$

NFA M_1



NFA M_2



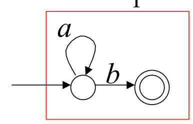
Single final state

Single final state

Example

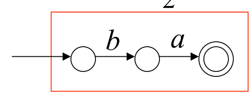
$L_1 = \{a^n b\}_{n \geq 0}$

M_1



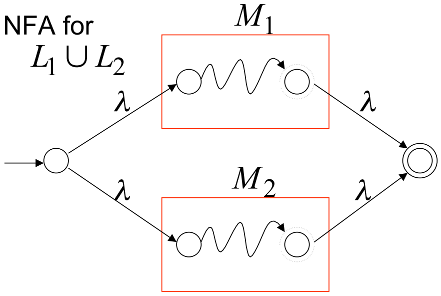
$L_2 = \{ba\}$

M_2



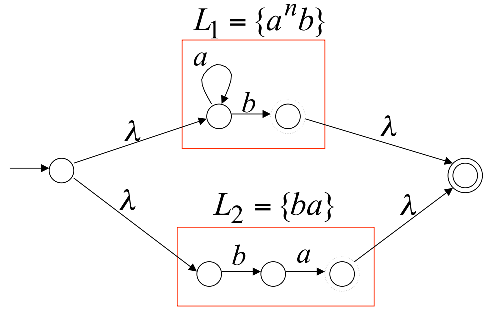
Union

- NFA for $L_1 \cup L_2$



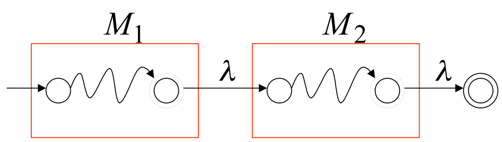
Example

NFA for $L_1 \cup L_2 = \{a^n b\} \cup \{ba\}$



Concatenation

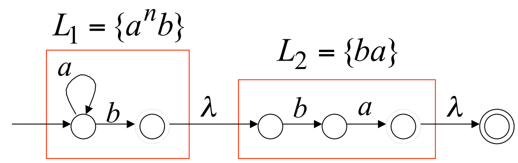
NFA for L_1L_2



Example

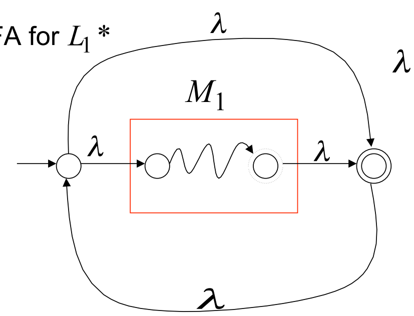
$L_1L_2 = \{a^n b\}\{ba\} = \{a^n bba\}$

NFA for



Star Operation

NFA for $L_1 *$



$\lambda \in L_1 *$

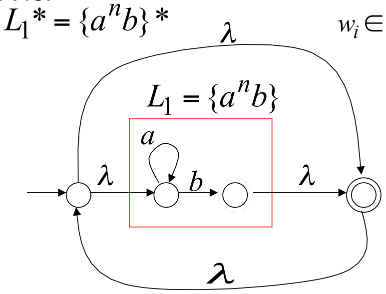
Example

NFA for

$L_1 * = \{a^n b\} *$

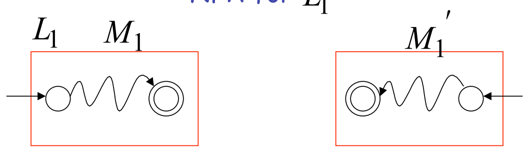
$w = w_1 w_2 \cdots w_k$

$w_i \in L_1$



Reverse

NFA for L_1^R

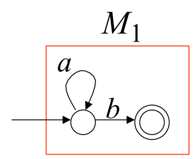


1. Reverse all transitions

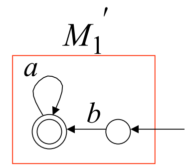
2. Make initial state final state
   and vice versa

Example

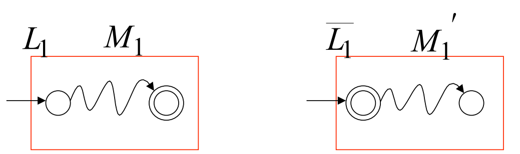
$L_1 = \{a^n b\}$



$L_1^R = \{ba^n\}$



Complement

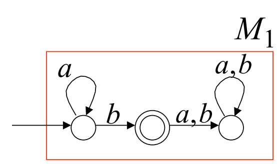


1. Take the **DFA** that accepts L_1

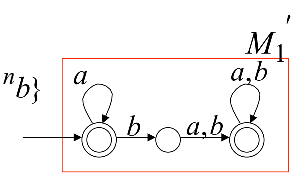
2. Make final states non-final,
   and vice-versa

Example

$L_1 = \{a^n b\}$



$\overline{L_1} = \{a,b\}^* - \{a^n b\}$



Intersection

DeMorgan's Law: $L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$

$L_1 \, , \; L_2$ regular

$\Longrightarrow$ $\overline{L_1} \, , \; \overline{L_2}$ regular

$\Longrightarrow$ $\overline{L_1} \cup \overline{L_2}$ regular

$\Longrightarrow$ $\overline{\overline{L_1} \cup \overline{L_2}}$ regular

$\Longrightarrow$ $L_1 \cap L_2$ regular

What's Next

- Read
  - Linz Chapter 1,2.1, 2.2. 2.3, (skip 2.4), 3, and Chapter 4
  - JFLAP Startup, Chapter 1, 2.1, (skip 2.2), 3, 4
- Next Lecture Topics from Chapter 4.2 and 4.3
  - Properties of regular languages
  - The pumping lemma (for regular languages)
- Quiz 1 in Recitation on Wednesday 9/17
  - Covers Linz 1.1, 1.2, 2.1, 2.2, 2.3, and JFLAP 1, 2.1
  - Closed book, but you may bring one sheet of 8.5 x 11 inch paper with any notes you like.
  - Quiz will take the full hour on Wednesday
- Homework
  - Homework Due Thursday