The Pumping Lemma:

• Given a infinite regular language $L$

• there exists an integer $m$

• for any string $w \in L$ with length $|w| \geq m$

• we can write $w = xyz$

• with $|xy| \leq m$ and $|y| \geq 1$

• such that $xy^iz \in L$ for $i = 0, 1, 2, ...$

Theorem: The language $L = \{a^n b^n : n \geq 0\}$ is not regular

Proof: Use the Pumping Lemma
Assume for contradiction that $L$ is a regular language.

Since $L$ is infinite, we can apply the Pumping Lemma.

Let $m$ be the integer in the Pumping Lemma.

Pick a string $w$ such that: $w \in L$, $|w| \geq m$.

We pick $w = a^m b^m$.

Write: $a^m b^m = xyz$

From the Pumping Lemma, it must be that length $|xy| \leq m$, $|y| \geq 1$.

Thus: $y = a^k$, $k \geq 1$.

$x y z = a^m b^m$

From the Pumping Lemma: $x y^i z \in L$, $i = 0, 1, 2, ...$

Thus: $x y^2 z \in L$.
From the Pumping Lemma: $x y z = a^m b^m, \ y = a^k, \ k \geq 1$  

Thus: $a^{m+k} b^m \in L$

BUT: $L = \{a^n b^n : n \geq 0\}$  

CONTRADICTION!!!

Therefore: Our assumption that $L$ is a regular language is not true

Conclusion: $L$ is not a regular language
More Applications
of
the Pumping Lemma

Non-regular languages
$L = \{vv^R : v \in \Sigma^*\}$

Regular languages

**Theorem:** The language

$L = \{vv^R : v \in \Sigma^*\}$   \(\Sigma = \{a, b\}\)

is not regular

**Proof:** Use the Pumping Lemma

$L = \{vv^R : v \in \Sigma^*\}$

Assume for contradiction that $L$ is a regular language

Since $L$ is infinite
we can apply the Pumping Lemma
\[ L = \{vv^R : v \in \Sigma^* \} \]

Let \( m \) be the integer in the Pumping Lemma

Pick a string \( w \) such that: \( w \in L \) and
\[ \text{length } |w| \geq m \]

We pick \( w = a^m b^m b^m a^m \)

Write \( a^m b^m b^m a^m = xyz \)

From the Pumping Lemma
it must be that length \( |xy| \leq m, \; |y| \geq 1 \)

\[ xyz = a\ldots a\ldots ab\ldots bb\ldots ba\ldots a \]
\[ x \quad y \quad z \]

Thus: \( y = a^k, \; k \geq 1 \)

\[ x \; y \; z = a^m b^m b^m a^m \]
\[ y = a^k, \; k \geq 1 \]

From the Pumping Lemma:
\[ x \; y^i \; z \in L \]
\[ i = 0, 1, 2, \ldots \]

Thus: \( x \; y^2 \; z \in L \)

\[ x \; y \; z = a^m b^m b^m a^m \]
\[ y = a^k, \; k \geq 1 \]

\[ x \; y^2 \; z \in L \]

From the Pumping Lemma:
\[ xy^2z = a\ldots a\ldots a\ldots ab\ldots bb\ldots ba\ldots a \in L \]
\[ x \quad y \quad y \quad z \]

Thus: \( a^{m+k} b^m b^m a^m \in L \)
\[ a^{m+k}b^mb^ma^m \in L \quad k \geq 1 \]

**BUT:** \[ L = \{vv^R : v \in \Sigma^* \} \]

\[ a^{m+k}b^mb^ma^m \notin L \]

**CONTRADICTION!!!**

Therefore: Our assumption that \( L \) is a regular language is not true

**Conclusion:** \( L \) is not a regular language

**Theorem:** The language
\[ L = \{a^n b^l c^{n+l} : n, l \geq 0\} \]

is not regular

**Proof:** Use the Pumping Lemma
\[ L = \{a^n b^l c^{n+l} : n, l \geq 0\} \]

Assume for contradiction that \( L \) is a regular language.

Since \( L \) is infinite, we can apply the Pumping Lemma.

Let \( m \) be the integer in the Pumping Lemma.

Pick a string \( w \) such that: \( w \in L \) and length \( |w| \geq m \).

We pick \( w = a^m b^m c^{2m} \).

Write \( a^m b^m c^{2m} = xyz \).

From the Pumping Lemma, it must be that length \( |xy| \leq m, |y| \geq 1 \).

Thus: \( xyz = a^k ab...bc...cc...c \).

Thus: \( y = a^k, k \geq 1 \).
From the Pumping Lemma: $xz \in L$

$xz = a^{m-k} a^m c^{2m} / x = a...aa...ab...bc...cc...c \in L$

Thus: $a^{m-k} b^m c^{2m} \in L$

BUT: $L = \{a^{n} b^l c^{n+l} : n, l \geq 0\}$

$a^{m-k} b^m c^{2m} \notin L$

CONTRADICTION!!!

Therefore: Our assumption that $L$ is a regular language is not true

Conclusion: $L$ is not a regular language

Non-regular languages $L = \{a^n : n \geq 0\}$

Regular languages
**Theorem:** The language $L = \{a^n : n \geq 0\}$ is not regular.

**Proof:** Use the Pumping Lemma.

$L = \{a^n : n \geq 0\}$

Assume for contradiction that $L$ is a regular language.

Since $L$ is infinite we can apply the Pumping Lemma.

Let $m$ be the integer in the Pumping Lemma.

Pick a string $w$ such that: $w \in L$ and $|w| \geq m$.

We pick $w = a^m$.

Write $a^m = xyz$.

From the Pumping Lemma it must be that length $|xy| \leq m$, $|y| \geq 1$.

Thus: $y = a^k$, $1 \leq k \leq m$.
From the Pumping Lemma:

\[ xy^i z \in L \quad i = 0, 1, 2, \ldots \]

Thus:

\[ xy^2 z \in L \]

From the Pumping Lemma:

\[ x y^2 z \in L \]

\[ xy^2 z = \underbrace{a\ldots aa\ldots aa\ldots aa\ldots aa\ldots a}_{m+k} \quad \underbrace{a}_{m!-m} \quad \underbrace{a}_{m!} \in L \]

Thus:

\[ a^{m!+k} \in L \]

Since:

\[ L = \{a^n : n \geq 0\} \]

There must exist \( p \) such that:

\[ m! + k = p! \]

However:

\[ m! + k \leq m! + m \quad \text{for} \quad m > 1 \]

\[ \leq m! + m! \]

\[ < m!m + m! \]

\[ = m!(m + 1) \]

\[ = (m + 1)! \]

\[ m! + k < (m + 1)! \]

\[ m! + k \neq p! \quad \text{for any} \quad p \]
\[ a^{m!+k} \in L \quad 1 \leq k \leq m \]

**BUT:** \[ L = \{a^n : n \geq 0\} \]

\[ a^{m!+k} \notin L \]

**CONTRADICTION!!!**

Therefore: Our assumption that \( L \) is a regular language is not true

**Conclusion:** \( L \) is not a regular language

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**What’s Next**

- Read
  - Linz Chapter 1, 2.1, 2.2, 2.3, (skip 2.4) 3, 4, 5.1-5.3
  - JFLAP Startup, Chapter 1, 2.1, 3, 4, 6.1
- Next Lecture Topics from Chapter 5.1-5.3
  - Context Free Grammars
- Quiz 1 in Recitation on Wednesday 9/17
  - Covers Linz 1.1, 1.2, 2.1, 2.2, 2.3 and JFLAP 1, 2.1
  - Closed book, but you may bring one sheet of 8.5 x 11 inch paper with any notes you like.
  - Quiz will take the full hour
- Homework
  - Homework Due Thursday