Polyhedral Compilation Foundations

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Overview of Today's Lecture

Outline:

- Follow-up on $\mathcal{Z}$-polyhedra
- Data dependence
  - Dependence representations
  - Various analysis
  - Data dependence algorithm in Candl/PoCC/Pluto

Mathematical concepts:

- Affine mapping
- Image, preimage by an affine mapping
- Cartesian product of polyhedra

Affine Function and Lattices (Reminder)

Definition (Affine function)

A function $f : \mathbb{K}^m \rightarrow \mathbb{K}^n$ is affine if there exists a vector $\vec{b} \in \mathbb{K}^n$ and a matrix $A \in \mathbb{K}^{m \times n}$ such that:

$$\forall \vec{x} \in \mathbb{K}^m, \ f(\vec{x}) = A\vec{x} + \vec{b}$$

Definition (Lattice)

A subset L in $\mathbb{Q}^n$ is a lattice if is generated by integral combination of finitely many vectors: $a_1, a_2, \ldots, a_n$ ($a_i \in \mathbb{Q}^n$).

$$L = L(a_1, \ldots, a_n) = \{\lambda_1 a_1 + \ldots + \lambda_n a_n \mid \lambda_i \in \mathbb{Z}\}$$

If the a_i vectors have integral coordinates, L is an integer lattice.

Example: $L_1 = \{2i+1, 3j+5 \mid i,j \in \mathbb{Z}\}$ is a lattice.

Image and Preimage

Definition (Image)

The image of a polyhedron $\mathcal{P} \in \mathbb{Z}^n$ by an affine function $f : \mathbb{Z}^n \rightarrow \mathbb{Z}^m$ is a $\mathcal{Z}$-polyhedron $\mathcal{P}'$:

$$\mathcal{P}' = \{f(\vec{x}) \in \mathbb{Z}^m \mid \vec{x} \in \mathcal{P}\}$$

Definition (Preimage)

The preimage of a polyhedron $\mathcal{P} \in \mathbb{Z}^n$ by an affine function $f : \mathbb{Z}^n \rightarrow \mathbb{Z}^m$ is a $\mathcal{Z}$-polyhedron $\mathcal{P}'$:

$$\mathcal{P}' = \{\vec{x} \in \mathbb{Z}^n \mid f(\vec{x}) \in \mathcal{P}\}$$

We have $Image(f^{-1}, \mathcal{P}) = Preimage(f, \mathcal{P})$ if f is invertible.

Relation Between Image, Preimage and $\mathcal{Z}$-polyhedra

- The image of a $\mathbb{Z}$-polyhedron by an unimodular function is a $\mathbb{Z}$-polyhedron
- The preimage of a $\mathbb{Z}$-polyhedron by an affine function is a $\mathbb{Z}$-polyhedron
- The image of a polyhedron by an affine invertible function is a $\mathcal{Z}$-polyhedron
- The preimage of a $\mathcal{Z}$-polyhedron by an affine function is a $\mathcal{Z}$-polyhedron
- The image by a non-invertible function is **not** a $\mathcal{Z}$-polyhedron

Returning to the Example

Exercise: Compute the set of cells of A accessed

Example

```
for (i = 0; i < N; ++i)
  for (j = i; j < N; ++j)
    A[2i + 3][4j] = i * j;
```

- $\mathcal{D}_S$: $\{i,j \mid 0 \leq i < N,\ i \leq j < N\}$
- Function: $f_A : \{2i+3, 4j \mid i,j \in \mathbb{Z}\}$
- $Image(f_A, \mathcal{D}_S)$ is the set of cells of A accessed (a $\mathcal{Z}$-polyhedron):
  - Polyhedron: $Q : \{i,j \mid 3 \leq i < 2N+2,\ 0 \leq j < 4N\}$
  - Lattice: $L : \{2i+3, 4j \mid i,j \in \mathbb{Z}\}$

Data Dependence

Definition (Bernstein conditions)

Given two references, there exists a dependence between them if the three following conditions hold:

- they reference the same array (cell)
- one of this access is a write
- the two associated statements are executed

Three categories of dependences:

- RAW (Read-After-Write, aka flow): first reference writes, second reads
- WAR (Write-After-Read, aka anti): first reference reads, second writes
- WAW (Write-After-Write, aka output): both references writes

Another kind: RAR (Read-After-Read dependences), used for locality/reuse computations

Purpose of Dependence Analysis

- Not all program transformations preserve the semantics
- Semantics is preserved if the dependence are preserved

- In standard frameworks, it means reordering statements
  - Statements containing dependent references should not be executed in a different order
  - Granularity: usually a reference to an **array**

- In the polyhedral framework, it means reordering statement **instances**
  - Statement instances containing dependent references should not be executed in a different order
  - Granularity: a reference to an **array cell**

Illustrations

Example

```
for (i = 0; i < N; ++i)
  for (j = 0; j < N; ++j)
    A[i][j] = A[i + N][j];
```

Example

```
for (i = 0; i < N; ++i)
  for (j = 0; j < N; ++j)
    A[i][j] = i * j;
for (i = 0; i < N; ++i)
  for (j = 0; j < N; ++j)
    B[i][j] = A[i][j];
```

An Intuitive Dependence Test Algorithm

Idea: compute the sets associated to the Bernstein conditions

Given two references a and b to the same array:

- Compute $\mathcal{W}_a$: $Image(f_a, \mathcal{D}_a)$ if a is a write, $\emptyset$ otherwise
- Compute $\mathcal{R}_a$: $Image(f_a, \mathcal{D}_a)$ if a is a read, $\emptyset$ otherwise
- Compute $\mathcal{W}_b$: $Image(f_b, \mathcal{D}_b)$ if b is a write, $\emptyset$ otherwise
- Compute $\mathcal{R}_b$: $Image(f_b, \mathcal{D}_b)$ if b is a read, $\emptyset$ otherwise

- If $\mathcal{W}_a \cap \mathcal{R}_b \neq \emptyset \vee \mathcal{W}_a \cap \mathcal{W}_b \neq \emptyset \vee \mathcal{R}_a \cap \mathcal{W}_b \neq \emptyset$ then $a\delta b$

A (Naive) Dependence Test Algorithm

Exercise: Write a dependence test algorithm for a program

A (Naive) Dependence Test Algorithm

Exercise: Write a dependence test algorithm for a program

- Create the Data Dependence Graph, with one node per statement
- For all pairs a, b of distinct references, do

  If a and b reference the same array, do

  (i) Compute $\mathcal{W}_a$, $\mathcal{R}_a$, $\mathcal{W}_b$, $\mathcal{R}_v$

  (ii) If $\mathcal{W}_a \cap \mathcal{R}_b \neq \emptyset \vee \mathcal{W}_a \cap \mathcal{W}_b \neq \emptyset \vee \mathcal{R}_a \cap \mathcal{W}_b \neq \emptyset$ then

  Add an edge between the statement with the reference a and the statement with the reference b in the DDG

Connection with Statement Instances

Objective: get the set of **instances** which are in dependence, not only statements

Exercise: Compute this set, from $\mathcal{W}_a$ and $\mathcal{R}_b$ (RAW dependence)

Connection with Statement Instances

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- Idea: $Preimage(f_a, \mathcal{W}_a \cap \mathcal{R}_b)$ gives the set of instances
- Must generalize to multiple references, we lose convexity (unions)

Some Terminology on Dependence Relations

We categorize the dependence relation in three kinds:

- **Uniform dependences:** the distance between two dependent iterations is a constant
  - ex: $i \rightarrow i+1$
  - ex: $i,j \rightarrow i+1,j+1$
- **Non-uniform dependences:** the distance between two dependent iterations varies along the execution
  - ex: $i \rightarrow i+j$
  - ex: $i \rightarrow 2i$
- **Parametric dependences:** at least a parameter is involved in the dependence relation
  - ex: $i \rightarrow i+N$
  - ex: $i+N \rightarrow j+M$

Data Dependence Analysis

Objective: compute the set of statement instances which are in dependence

Some of the several possible approaches:

- Compute the transitive closure of the access function
  - Problems: transitive closure is not convex in general, and not even computable in many situations
- Compute an indicator of the distance between two dependent iterations
  - Problems: approximative for non-uniform dependences
- dependence cone: do the union of dependence relations
  - Problems: over-approximative as it requires union and transitive closure to model all dependences in a single cone
- Retained solution: **dependence polyhedron**, list of sets of dependent instances

Dependence Polyhedron [1/3]

Principle: model all **pairs** of instances in dependence

Definition (Dependence of statement instances)

A statement S depends on a statement R (written $R \rightarrow S$) if there exists an operation $S(\vec{x}_S)$ and $R(\vec{x}_R)$ and a memory location m such that:

1. $S(\vec{x}_S)$ and $R(\vec{x}_R)$ refer to the same memory location m, and at least one of them writes to that location,
2. x_S and x_R belongs to the iteration domain of R and S,
3. in the original sequential order, $S(\vec{x}_S)$ is executed before $R(\vec{x}_R)$.

Dependence Polyhedron [2/3]

1. *Same memory location*: equality of the subscript functions of a pair of references to the same array: $F_S\vec{x}_S + a_S = F_R\vec{x}_R + a_R$.

2. *Iteration domains*: both S and R iteration domains can be described using affine inequalities, respectively $A_S\vec{x}_S + c_S \geq 0$ and $A_R\vec{x}_R + c_R \geq 0$.

3. *Precedence order*: each case corresponds to a common loop depth, and is called a *dependence level*.

   For each dependence level l, the precedence constraints are the equality of the loop index variables at depth lesser to l: $x_{R,i} = x_{S,i}$ for $i < l$ and $x_{R,l} > x_{S,l}$ if l is less than the common nesting loop level. Otherwise, there is no additional constraint and the dependence only exists if S is textually before R.

   Such constraints can be written using linear inequalities: $P_{l,S}\vec{x}_S - P_{l,R}\vec{x}_R + b \geq 0$.

Dependence Polyhedron [3/3]

The dependence polyhedron for $R \to S$ at a given level l and for a given pair of references f_R, f_S is described as [Feautrier/Bastoul]:

$$\mathcal{D}_{R,S,f_R,f_S,l} : D\begin{pmatrix} \vec{x}_S \\ \vec{x}_R \end{pmatrix} + d = \left[\begin{array}{cc} F_S & -F_R \\ \hline A_S & 0 \\ 0 & A_R \\ PS & -P_R \end{array}\right] \begin{pmatrix} \vec{x}_S \\ \vec{x}_R \end{pmatrix} + \begin{pmatrix} a_S - a_R \\ c_S \\ c_R \\ b \end{pmatrix} \begin{array}{c} = 0 \\ \hline \geq \vec{0} \end{array}$$

A few properties:

- We can always build the dep polyhedron for a given pair of affine array accesses (it is convex)
- It is exact, if the iteration domain and the access functions are also exact
- it is over-approximated if the iteration domain or the access function is an approximation

Polyhedral Representation of Programs

Static Control Parts

- Loops have affine control only (over-approximation otherwise)

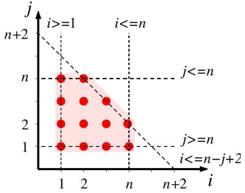
Polyhedral Representation of Programs

Static Control Parts

- Loops have affine control only (over-approximation otherwise)
- Iteration domain: represented as integer polyhedra

```
for (i=1; i<=n; ++i)
. for (j=1; j<=n; ++j)
. . if (i<=n-j+2)
. . . s[i] = ...
```

$$\mathcal{D}_{S1} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ -1 & -1 & 1 & 2 \end{bmatrix} . \begin{pmatrix} i \\ j \\ n \\ 1 \end{pmatrix} \geq \vec{0}$$



Iteration domain of S_1

Polyhedral Representation of Programs

Static Control Parts

- Loops have affine control only (over-approximation otherwise)
- Iteration domain: represented as integer polyhedra
- Memory accesses: static references, represented as affine functions of $\vec{x_S}$ and $\vec{p}$

```
for (i=0; i<n; ++i) {
. s[i] = 0;
. for (j=0; j<n; ++j)
. . s[i] = s[i]+a[i][j]*x[j];
}
```

$$f_{\mathbf{s}}(\vec{x_{S2}}) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} . \begin{pmatrix} \vec{x_{S2}} \\ n \\ 1 \end{pmatrix}$$

$$f_{\mathbf{a}}(\vec{x_{S2}}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} . \begin{pmatrix} \vec{x_{S2}} \\ n \\ 1 \end{pmatrix}$$

$$f_{\mathbf{x}}(\vec{x_{S2}}) = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} . \begin{pmatrix} \vec{x_{S2}} \\ n \\ 1 \end{pmatrix}$$

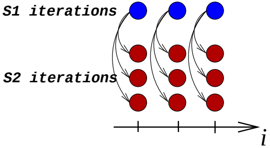
Polyhedral Representation of Programs

Static Control Parts

- Loops have affine control only (over-approximation otherwise)
- Iteration domain: represented as integer polyhedra
- Memory accesses: static references, represented as affine functions of $\vec{x}_S$ and $\vec{p}$
- Data dependence between S1 and S2: a subset of the Cartesian product of $\mathcal{D}_{S1}$ and $\mathcal{D}_{S2}$ (**exact analysis**)

```
for (i=1; i<=3; ++i) {
. s[i] = 0;
. for (j=1; j<=3; ++j)
. . s[i] = s[i] + 1;
}
```

$$\mathcal{D}_{S1\delta S2}: \left[\begin{array}{cccc} 1 & -1 & 0 & 0 \\ \hline 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 3 \end{array}\right] . \begin{pmatrix} i_{S1} \\ i_{S2} \\ j_{S2} \\ 1 \end{pmatrix} \begin{array}{c} = 0 \\ \geq \vec{0} \end{array}$$



A Dependence Polyhedra Construction Algorithm

1 Initialize reduced dependence graph with one node per program statement
2 For each pairs of statements R, S do
3   For each pairs of distinct references f_R, f_S to the same array, do
4   If R, S does not share any loop, $min\_depth = 0$ else $min\_depth = 1$
5   For each level l from min_depth to nb_common_loops, do
6   Build the dependence polyhedron $\mathcal{D}_{R,S,f_R,f_S,l}$
7   If $\mathcal{D}_{R,S,f_R,f_S,l} \neq \emptyset$ then
8   If f_R is a write and f_S is a read, $type = RAW$
9   If f_R is a write and f_S is a write, $type = WAW$
10   If f_R is a read and f_S is a write, $type = WAR$
11   If f_R is a read and f_S is a read, $type = RAR$
12                 $add\_edge(R, S, \{l, \mathcal{D}_{R,S,f_R,f_S,l}, type\})$

The PolyLib Matrix Format

All our tools use this notation (Candl, Pluto, Cloog, PIPLib, etc.)

Given $\mathcal{D}_{R,S}$: $\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix} . \begin{pmatrix} i_R \\ i_S \\ j_S \\ n \\ 1 \end{pmatrix} \quad \frac{=0}{\geq \vec{0}}$

It is written: $\begin{bmatrix} \mathbf{0} & 1 & -1 & 0 & 0 & 0 \\ \mathbf{1} & 1 & 0 & 0 & 0 & 0 \\ \mathbf{1} & -1 & 0 & 0 & 1 & 0 \\ \mathbf{1} & 0 & 1 & 0 & 0 & 0 \\ \mathbf{1} & 0 & -1 & 0 & 1 & 0 \\ \mathbf{1} & 0 & 0 & 1 & 0 & 0 \\ \mathbf{1} & 0 & 0 & -1 & 1 & 0 \end{bmatrix} . \begin{pmatrix} i_R \\ i_S \\ j_S \\ n \\ 1 \end{pmatrix}$

On the first column, 0 stands for $= 0$, 1 for ≥ 0

Practicing

Exercise: Give all dependence polyhedra

Example

```
for (i = 0; i < N; ++i)
  for (j = 0; j < N; ++j)
    A[i][j] = A[i + 1][j + 1];
```

Example

```
for (i = 0; i < N; ++i)
  for (j = 0; j < N; ++j)
    A[i][j] = i * j;
for (i = 0; i < N; ++i)
  for (j = 0; j < N; ++j)
    B[i][j] = A[i][j];
```

Connection with Parallelism

- A dependence is loop-carried if 2 iterations of this loop access the same array cell
- If no such dependence exists, the loop is parallel
- A parallel loop can be transformed arbitrarily
- OpenMP free parallelization or vectorization is possible

Example

```
for (i = 0; i < N; ++i)
  for (j = 0; j < N; ++j)
    C[i][j] = 0;
for (i = 0; i < N; ++i)
  for (j = 0; j < N; ++j)
    for (k = 0; k < N; ++k)
      C[i][j] += A[i][k] * B[k][j];
```

Practicing Parallelism

Exercise: Give all parallel loops

Example

```
for (i = 0; i < N; ++i)
  for (j = 0; j < N; ++j)
    A[i][j] = i * j;
for (i = 0; i < N; ++i)
  for (j = 0; j < N; ++j)
    B[i][j] = A[i][j];
```

Example

```
for (t = 0; t < L; ++t)
  for (j = 1; j < N - 1; ++j)
    A[j] = A[j - 1] + A[j] + A[j + 1];
```

Visual Intuition

- Synchronization-free parallelism means "slices" in the dependence polyhedron
- The shape of the independent slices gives an intuition of which loop of the program are parallel
- Transforming the code may expose (more) parallelism possibilities
- Be careful of multiple references: must do the union of the dependence relations

Other Techniques for Dependence Analysis

- Scalars are a particular case of array (c = c[0])
- **Privatization:** a variable is written before it is read (use-def chains)
- **Renaming:** two privatized variables having the same name
- **Expansion:** remove dependences by increasing the array dimension
- Transform program to Single-Assignment-Form (SSA)

- Scalar privatization / renaming / expansion is implemented in Candl
- Maximal static expansion is efficient but difficult!

Hands On!

Demo of Clan + Candl

A First Intuition About Scheduling

Intuition: the source iteration must be executed before the target iteration

Definition (Precedence condition)

Given Θ_R a schedule for the instances of R, Θ_S a schedule for the instances of S. Then, $\forall \langle \vec{x}_R, \vec{x}_S \rangle \in \mathcal{D}_{R,S}$:

$$\Theta_R(\vec{x}_R) \prec \Theta_S(\vec{x}_S)$$

Next week: scheduling and semantics preservation (Farkas method, convex space of legal schedules, tiling hyperplane method)