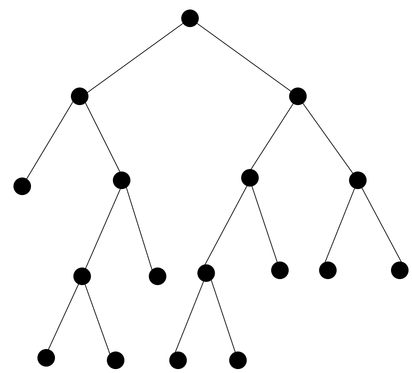
Homework 1, CS 301, McConnell, Spring '09, due Tues. Feb. 3

Modified 1/25 20:25

Modified 2/3 15:15 to correct a typo on a problem that will be carried over to Homework 2.

February 4 is the end of the add/drop period.

1. A *full binary tree* is a rooted tree where every internal (non-leaf) node has exactly two children.



Show by induction on the number of leaves that the number of leaves in a full binary tree is always one greater than the number of internal nodes. (The drawing illustrates this: it has nine leaves and eight internal nodes.)

I can think of two alternative induction proofs. I will give you the first. Fill in the details of the second, to complete your own proof, using the hints.

Here's the first proof. Let T be a full binary tree with n leaves.

Base case: If $n = 1$, then the tree has one leaf and zero internal nodes, and the claim holds.

Induction step: Suppose $n > 1$ and the claim holds for for full binary trees with fewer than n leaves.

Let T be an arbitrary full binary tree with n leaves. Remove the root of T to get two smaller full binary trees that the induction hypothesis applies to. They each have one fewer leaves than internal nodes, and together they have n leaves. They must have $n-2$ internal nodes between them. Adding the root back in, we have that T has $n-1$ internal nodes, and the claim holds. Since T is an arbitrary full binary tree with n leaves, the claim holds for all full binary trees with n leaves.

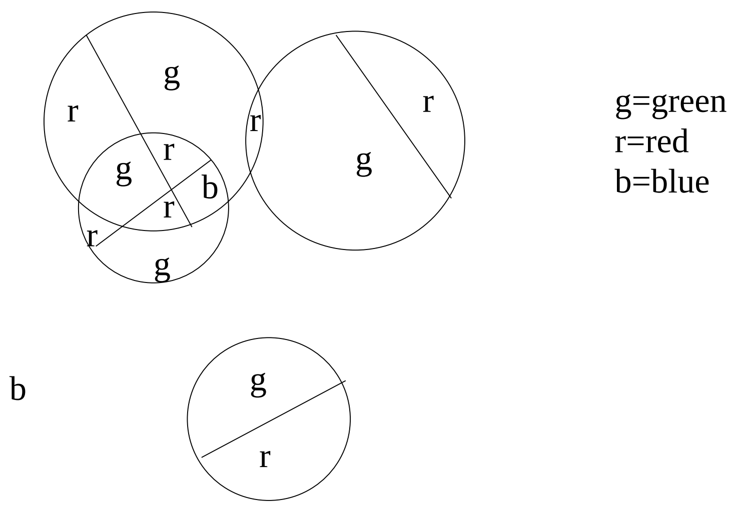
For your proof, make use of a different strategy for the induction step. The deepest nodes of T come in sibling pairs. Remove two leaf siblings from T to get a smaller full binary tree T'. Apply the induction hypothesis to T', and show that this implies that the claim holds for T.

2. Here's a proof that when you lay down n lines in the plane, you can color the resulting regions with two colors so that no two regions that share a border have the same color. (Two regions that share only a corner can be colored with the same color.)

As a base case, if $n = 1$, you can color the plane green on one side of the line and red on the other.

For the induction step, suppose $n > 1$ and assume that the claim holds for any $n - 1$ lines in the plane. Consider an arbitrary arrangement of n lines in the plane. Remove one of the lines. By the induction hypothesis, there exists a legal way to color the regions red and green. Add the removed line back in. Now, two regions in the result violate the rules if and only if they share a border on the new line. On one side of this line, change every red region to green and every green region to red. This resolves the violations without creating any new ones, so it gives a legal coloring of the original arrangement.

Using this proof as a model, show that when you lay down a bunch of circles, each with a chord, in the plane, you can always color the regions with three colors so that no two regions sharing a border have the same color:



3. Let $X = \{1, 2, 3, 4\}$ and $Y = \{0, 2, 4, 6\}$. List the elements in the following sets:

   (a) $X \cup Y$

   (b) $X \cap Y$

   (c) $X - Y$

   (d) The **power set** $\mathcal{P}(X)$ of X, which is, the set of all subsets of X. Don't forget X and the empty set.

4. Show that the size of the power set of a set with n elements is 2^n. You can either use induction or the *product rule* from your discrete math class.

5. Let $X = \{a, b\}$ and $Y = \{1, 2, 3\}$.

   (a) List the elements of $X \times Y$.

   (b) A **binary relation** from X to Y is any subset of $X \times Y$. How many different binary relations from X to Y are there? *Hint: something you've shown in a previous problem is helpful.*

(c) A **function** from **domain** X to **codomain** Y is any subset of $X \times Y$ such that each element of X appears exactly once on the left side of an element. It is a special case of a binary relation, since it meets the original requirements, plus some additional ones. List the functions from X to Y. (Each function should be a set of ordered pairs.)

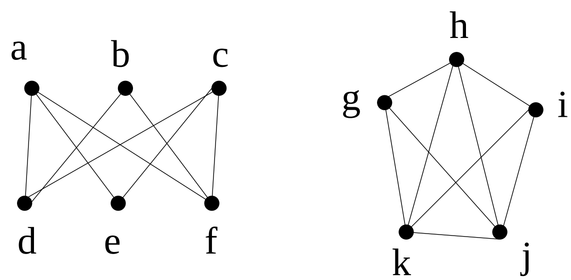
6. Look up the definition of an **equivalence relation** in Chapter 1.

   (a) The set of pairs $\{(i, j)|i$ and j have the same remainder when divided by 11$\}$ is obviously a relation from the set of integers to the set of integers. Show that it is an equivalence relation by explaining why it is reflexive, symmetric, and transitive.

   (b) How many equivalence classes does this relation have?

   (c) Let G be an undirected graph on vertex set V. Show that the following is an equivalence relation on V: $\{(u, v)|$there is a path from u to $v\}$.

   (d) What are the equivalence classes that this relation gives in the following graph?



   (e) Here is a relation on the *pairs* of integers: $\{([m, n], [i, j])|m + j = n + i\}$. Show that this is an equivalence relation. **The original posting of this problem had a typo that changes the problem. It is still possible to prove the changed claim, but I will carry this version over to Homework 2.**

7. The *subset* relation on the power set of X is the set of pairs of the form $\{(Y, Z)|Y \subseteq Z\}$.

   (a) List the elements of the subset relation on the power set of $X = \{1, 2, 3\}$.

   (b) A relation R is **antisymmetric** if, whenever (a, b) and (b, a) are elements of R, then $a = b$. Argue that the subset relation is antisymmetric.

   (c) A **partial order** relation is a relation from a set S to itself that is reflexive, antisymmetric, and transitive. Show that the subset relation is a partial order on the power set of X. (A partial order is sometimes called a **poset**, which stands for "partially ordered set.")

   (d) Let X be the articles of clothing Frank puts on in the morning. Suppose $X = \{shirt, pants, boots, undershorts, socks, wristwatch, T{-}shirt, jacket, mittens, cap\}$. For $y, z \in X$, consider the relation $R = \{(y, z)|y$ must be put on no later than $z\}$. Examples of elements in R are $(socks, boots)$ and $(undershorts, pants)$ and $(pants, boots)$. Show that R is a partial order.

   (This explains the term *partial order*: it places constraints on the ways tasks can be ordered, but doesn't necessarily force a unique ordering.)

8. A **bijection** from X to Y is a special case of a function with domain X and codomain Y: it meets the additional requirement that every element from Y occurs exactly once on the righthand side of an ordered pair.

   If $X = \{a, b, c\}$ and $Y = \{1, 2, 3\}$, then $\{(a, 3), (b, 1), (c, 2)\}$ is a bijection, but $\{(a, 3), (b, 1), (c, 3)\}$ is not.

   Note that a bijection can only exist if $|X| = |Y|$: if all the males at a dance have a female dance partner and vice versa, then the number of males at the dance must be equal to the number of females.

   Therefore, finding a bijection from one set to another is a way to prove that the two sets have the same cardinality.

   For example, if $|Z|$ is odd, we can show that it has the same number of even-sized subsets as odd-sized subsets by observing that $\{(X, Z - X) | X \subseteq Z \text{ and } |X| \text{ is odd}\}$ is a bijection from odd-sized subsets to even-sized subsets.

   (a) Why doesn't this proof work when $|Z|$ is even?

   (b) Nevertheless, playing with some examples shows that the claim still appears to be true when $|Z|$ is even.

   Prove that it's true whether $|Z|$ is even or odd by using a different bijection. *Hint: Suppose Z is the people in our class, and divide the odd sized subsets into those that don't contain Frank and those that do. Each of these sets can be matched with an even-sized subset in a natural way.*

9. We said that an infinite set is countably infinite if there is a way to order the elements so that each element appears in the resulting list. This is just another way of saying that there is a bijection between the elements and the natural numbers: each element is paired with the number that gives its position in the list.

   To show that an infinite set is not countably infinite, it is not enough to give an ordering in which to list its elements so that some elements don't appear. For example, we saw a list of all finite strings in alphabetic order starts off with the null string, then a, then aa, etc., and never gets around to any string with a b in it. This didn't prove that the set of all finite strings is uncountably infinite, just that we had picked a stupid order to list them in. By listing them in ascending order of length, we saw that the list could be made to contain every string.

   Let's now consider the set of *infinite* strings on the English alphabet. Attempts to find a way to order them so that none is missed seem to fail. Show that the problem was not that we are too stupid, but that no such order exists. Use the technique of proof by contradiction: assume that such an order exists (possibly known only by an omniscient deity), and show a contradiction.

   (Note: We have seen two kinds of infinity, where one is greater than the other. This discovery was made by Georg Cantor, who discovered that there is an infinite hierarchy of infinities, each one dwarfed by the next. Cantor spent much of the rest of his life in an insane asylum.)