Theoretical Computer Science 132 (1994) 209–227
Elsevier

A k-structure generalization of the theory of 2-structures

A. Ehrenfeucht and R. McConnell

Department of Computer Science, University of Colorado, Campus Box 430, Boulder, CO 80309-0430, USA

Communicated by A. Salomaa
Received March 1993
Revised August 1993

Abstract

Ehrenfeucht. A. and R. McConnell, A k-structure generalization of the theory of 2-structures, Theoretical Computer Science 132 (1994) 209–227.

The *prime tree decomposition* of graphs facilitates the solution of a number of important combinatorial problems. The decomposition is also known as *modular decomposition* and *substitution decomposition.* It has been generalized to 2-structures and to k-ary relations, but while both of these generalizations give the decomposition on graphs as a special case, neither is a generalization of the other. In this paper, we propose a type of edge-colored hypergraph, which we will call a k-structure. We define the modular decomposition of k-structures, and generalize the essential algebraic properties. This unifies the prime tree decompositions on 2-structures and k-ary relations, by giving them as special cases, and extends the decomposition to k-structures that are neither 2-structures nor k-ary relations. In addition, we show that any indecomposable k-structure on $n \geqslant 3$ nodes contains a smaller indecomposable substructure that has at least $n-k$ nodes. This is a generalization of a previously known result on graphs and 2-structures, that is, where $k=2$. The generalization to k-ary relations that it gives as a special case is also new.

1. Introduction

Given a finite graph G, suppose there is a partition $\mathbf{P}$ of the nodes of G such that for any two distinct partition classes $X, Y \in \mathbf{P}$, either every element of X is adjacent to every element of Y, or no element of X is adjacent to any element of Y. Such a partition of G is known as a *congruence partition.* A *factor* is the subgraph induced in G by a partition class in $\mathbf{P}$. The *quotient graph*, $G/\mathbf{P}$, is a graph whose nodes are the partition classes of $\mathbf{P}$: If $X, Y \in \mathbf{P}$ and $X \neq Y$, then (X, Y) is an edge in $G/\mathbf{P}$ if the

Correspondence to: R. McConnell, Computer Science Department, University of Colorado, Boulder, CO 80309, USA. Email: ross@cs.colorado.edu.

SSDI 0304-3975(93)E0156-X

members of Y are adjacent to those of X. The quotient and the factors corresponding to its nodes uniquely specify G.

This decomposition is not unique; there may be many congruence partitions on a graph. In addition, it is possible that the quotient and the factors may themselves be decomposed recursively, giving a nested decomposition. A set of fundamental properties have been proven about the congruence partitions and their partition classes. From these properties one may derive a unique nested decomposition of the graph, called the *prime tree family* [9, 10], which represents all congruence partitions.

The prime tree family was first described by Gallai [12], and has since been rediscovered independently by different researchers. The decomposition is also known as the *substitution decomposition* [19], *modular decomposition* [23], and the *X-join decomposition* [14]. The partition classes in the congruence partitions are known as *clans* [9], *closed sets* [12], *modules* [23], *autonomous sets* [19], *partitive sets* [13], *clumps* [1], and *stable sets* [25]. The series-parallel decomposition of general series-parallel orders [32] is a special case of this decomposition. The decomposition is also closely related to the *split decomposition* of Cunningham and Edmonds [6]. Kelly [16] gives a survey of the history of the idea.

There is a large number of combinatorial problems on graphs and partial orders whose solution may be facilitated by the prime tree family of the graph or partial order. Examples include finding maximum-weight cliques and maximum-weight matchings, minimum node colorings [2], finding the dimension of a partial order [29], constructing perfect graphs [2], finding whether a graph is a comparability graph [28], and solving certain scheduling problems [26]. Möhring [19] gives a review.

The prime tree family has also been generalized to infinite graphs and other structures of infinite size [22, 15], but in this paper, we deal only with structures of finite size. Generalization of the decomposition to other finite structures besides graphs has proceeded in two directions. The first generalization of the decomposition is to *2-structures* [9, 10]. If D is a set, $E_2(D)$ denotes the set $\{(x, y), x, y \in D \text{ and } x \neq y\}$. A 2-structure [9] is a set D and a partition, or "coloring", of the members of $E_2(D)$. A directed graph is a special case of a 2-structure, since it is a partition of $E_2(D)$ into those members that are edges in the graph and those that are not. An undirected graph is a special case of a directed graph, where whenever (x, y) is a directed edge, (y, x) is also an edge. Ehrenfeucht and Rozenberg [9, 10] give a generalization of the notion of the congruence partition to 2-structures and prove the fundamental properties for the generalization.

The second generalization has been to k-ary relations. If D is a set, a k-ary relation on D is a subset of D^k. A graph is a special case where $k=2$: D is the set of nodes and the edges of the graph are a subset of $D \times D$. Möhring and Radermacher [22] give the generalization of the decomposition to k-ary relations and prove the fundamental theorems. The fundamental theorems on either k-ary relations or on 2-structures give the analogous theorems on graphs as a special case. Reflexive elements of the form $(x, x, \ldots, x)$ are irrelevant to the congruence partitions, so for this paper we will assume without loss of generality that a k-ary relation is antireflexive. The generalization of the prime tree family to k-ary relations has applications in the decomposition of nondeterministic automata [30, 31].

Although 2-structures and k-ary relations are both generalizations of the notion of a graph, neither is a generalization of the other. In this paper, we propose a generalization of 2-structures and their congruence partitions to structures called *k-structures*. Antireflexive k-ary relations and 2-structures are special cases of k-structures. In addition, there are k-structures that are not k-ary relations or 2-structures, so the generalization is a proper one. We give theorems on k-structures that give the fundamental properties of congruence partitions on 2-structures and antireflexive k-ary relations as special cases. Our approach is to show that congruence partitions on k-structures satisfy the requirements of the general algebraic model for congruence partitions developed by Möhring and Radermacher [22] on previous structures.

We then give a result on *primitive k-structures* that is new also for k-ary relations. Primitive k-structures are k-structures that have only trivial congruence partitions. In the cases of graphs and 2-structures, these are known as *primitive* [9] or *prime* [19]. The maximum size of primitive subgraph is the critical element for the usefulness of the decomposition in many of the graph algorithms that make use of the decomposition. Ehrenfeucht and Rozenberg [11] show that every primitive 2-structure on $n>2$ nodes contains a primitive substructure that has at least $n-2$ nodes. This establishes the existence of a secondary decomposition of primitive structures in terms of a recursive series of smaller primitive structures. We show that every primitive k-structure has a primitive substructure that has at least $n-k$ nodes, establishing a similar decomposition for primitive k-structures.

2. Preliminaries

Let ϕ be a mapping from a set X onto a set Y. Then $\phi(x)$ denotes the element of Y to which ϕ maps x. If $X' \subseteq X$, then $\phi(X')$ denotes the set $\{y\colon y=\phi(x)$ for some $x \in X'\}$. Let $e=(x_1, x_2, \ldots, x_k)$ be a sequence of not necessarily distinct members of X. Then $\phi(e)$ denotes $(\phi(x_1), \phi(x_2), \ldots, \phi(x_k))$. The *image* of x, X', and e denotes $\phi(x)$, $\phi(X')$, and $\phi(e)$, respectively. If $y \in Y$, then $\phi^{-1}(y)$ denotes the set $\{x\colon \phi(x)=y\}$. If $Y' \subseteq Y$, then $\phi^{-1}(Y')$ denotes the set $\{x\colon \phi(x) \in Y')$. If $e=(y_1, y_2, \ldots, y_k)$ is a sequence of not necessarily distinct elements of Y, then $\phi^{-1}(e)$ denotes the set of k-tuples given by $(\{x_1, x_2, \ldots, x_k)\colon \phi(x_i)=y_i$ for each $i \in \{1, 2, \ldots, k\}\}$. The *inverse image* of y, Y', and e denotes $\phi^{-1}(y)$, $\phi^{-1}(Y')$, and $\phi^{-1}(e)$, respectively.

Let D be a set. A family $\mathbf{P}$ of subsets of D is a *partition of* D if $\bigcup \mathbf{P}=D$, and for every pair $P_1, P_2 \in \mathbf{P}$ such that $P_1 \neq P_2$, $P_1 \cap P_2 = \emptyset$. $\mathbf{P}$ defines a mapping $\phi_{\mathbf{P}}$ of D onto $\mathbf{P}$: $\phi_{\mathbf{P}}(x)=P$, where P is the member of $\mathbf{P}$ that contains x. A *system of representatives of* $\mathbf{P}$ is a set S such that $S \subseteq \bigcup \mathbf{P}$ and for any $P \in \mathbf{P}$, $P \cap S = 1$. A set D is a *singleton set* if $|D|=1$.

Let X and Y be two sets. X and Y *overlap* if $X-Y$, $X \cap Y$ and $Y-X$ are all nonempty.

Definition. A family $\mathbf{F}$ of sets is a *point-partitive hypergraph* [4] if it has the following properties

(1) $\bigcup \mathbf{F} \in \mathbf{F}$, and the singleton subsets of $\bigcup \mathbf{F}$ are each members of $\mathbf{F}$.

(2) For any pair $X, Y \in \mathbf{F}$, if X and Y overlap, then $X-Y$, $Y-X$, $X \cap Y$ and $X \cup Y$ are also members of $\mathbf{F}$.

Theorem 2.1 [22]. *Let* $\mathbf{F}$ *be a point-partitive hypergraph, and let* $\mathbf{F}'$ *be the members of* $\mathbf{F}$ *that overlap no members of* $\mathbf{F}$. *If* $X \in \mathbf{F}'$, *let parent*(X) *denote the minimum-cardinality* $Y \in \mathbf{F}'$ *such that* $X \subset Y$.

(1) *The parent relation defines a tree. If* $Y \in \mathbf{F}'$ *and* $|Y| > 1$, *then the children of* Y *in* T *are in a partition of* X.

(2) *Every member of* $\mathbf{F}$ *is a union of siblings in* T. *Each nonleaf member* X *of* $\mathbf{F}'$ *has one of the following properties*

(a) X *is* q-complete: *the union of any subfamily of its children is a member of* $\mathbf{F}$;

(b) X *is* q-primitive: *Other than* X *and each of its children, no union of a subfamily of its children is a member of* $\mathbf{F}$.

(c) X *is* q-linear: *There is a linear ordering of children of* X *such that a union of a subfamily of its children is a member of* $\mathbf{F}$ *if and only if those children are consecutive in the linear order.*

Definition. If $\mathbf{F}$ is a point-partitive hypergraph, a set $X \in \mathbf{F}$ is *prime* in $\mathbf{F}$ if it overlaps no other member of $\mathbf{F}$. The *prime tree representation* of $\mathbf{F}$ is the following: Let $\mathbf{F}'$ be the family of prime members of $\mathbf{F}$. Label each member of $\mathbf{F}'$ as q-complete, q-primitive, or q-linear, in accordance with Theorem 2.1. Supply pointers from each member X of $\mathbf{F}'$ to its children, i.e., to the maximal-cardinality members of $\mathbf{F}'$ that are proper subsets of X. If X is q-linear, supply the linear ordering of its children specified by Theorem 2.1.

Using the rules given by Theorem 2.1, it is then trivial to determine whether any subset of $\bigcup \mathbf{F}$ is a member of $\mathbf{F}$. One advantage of this representation is that it represents $\mathbf{F}$ in $O(n)$ space, where $n = |\bigcup \mathbf{F}|$, even though $\mathbf{F}$ may have as many as 2^n members.

3. k-structures and their prime tree families

Definition. Let D be a finite set. A sequence $(x_1, x_2, \ldots, x_k)$ is a *heterogeneous sequence on D of length k* if there exist $i, j \in \{1, 2, \ldots, k\}$ such that $x_i \neq x_j$. $E_k(D)$ is the set of heterogeneous sequences on D of length k. Following the approach used for 2-structures in [8], we define a *k-structure* on domain D to be a function $g: E_k(D) \rightarrow \Delta$, where Δ is a given set of labels. For notational convenience, if $e \in E_k(D)$, we will sometimes call $g(e)$ the *color* of e.

If g is k-structure on domain D, the elements of D are its nodes. D will be denoted $dom(g)$, and the elements of $E_k(dom(g))$ are the *edges* of g.

Note that when $k=2$, this definition is equivalent to the definition of a 2-structure. An antireflexive k-ary relation on D is a subset R of $E_k(D)$. Alternatively, it may be viewed as a k-structure $E_k(D) \to \{0,1\}$, where $g(e)=1$ iff $e \in R$. Thus, results about k-structures apply both to 2-structures and to k-ary relations.

Definition. Suppose g is a k-structure and $e=(x_1, x_2, \dots, x_k)$ is an edge of g. Then the *support of* e, denoted $supp(e)$, is the set $\{x: i \in \{1,2,\dots,k\}\}$. For a given i, x_i is the *ith projection* of e, and is denoted $\pi_i(e)$.

Definition. If g is a k-structure $E_k(dom(g)) \to \Delta$ and $X \subseteq dom(g)$, then the *substructure induced in g by X*, denoted $g \mid X$, is the function $E_k(X) \to \Delta$, such that whenever $e \in E_k(X)$, $(g \mid X)(e) = g(e)$. That is, it is the coloring of the elements $E_k(X)$ that is given by their coloring in g. $g \mid X$ is *proper substructure* of g if X is a proper subset of $dom(g)$.

Definition. Let g be a k-structure $E_k(dom(g)) \to \Delta$, and let $\mathbf{P}$ be a partition of $dom(g)$. $\mathbf{P}$ is a set family, so each member of $E_k(\mathbf{P})$ is a sequence of sets. $\mathbf{P}$ is a *congruence partition* on g if for each $e \in E_k(\mathbf{P})$, and each $e, f \in \phi_{\mathbf{P}}^{-1}(\mathbf{e})$, $g(e)=g(f)$. That is, it is a congruence partition if all members of $\phi_{\mathbf{P}}^{-1}(\mathbf{e})$ are the same color in g. In this case, the quotient $g/\mathbf{P}$ is the k-structure $E_k(\mathbf{P}) \to \Delta$, where $(g/\mathbf{P})\ (\phi_{\mathbf{P}}(e)) = g(e)$ for any $e \in E_k(dom(g))$. That is, $g/\mathbf{P}$ is the k-structure whose domain is $\mathbf{P}$, where the color of each member of $E_k(\mathbf{P})$ is given by the color of the edges in its inverse image. The *factors* are the substructures indicated in g by members of $\mathbf{P}$.

The quotient $g/\mathbf{P}$ and the factors $g \mid P$ for $P \in \mathbf{P}$ determine g uniquely. Indeed, let $e \in E_k(dom(g))$. If $\phi_{\mathbf{P}}(e) \in E_k(\mathbf{P})$, then $g(e) = (g/\mathbf{P})(\phi_{\mathbf{P}}(\mathbf{e}))$. On the other hand, if $\phi_{\mathbf{P}}(e) \notin E_k(\mathbf{P})$, i.e. if $\phi_{\mathbf{P}}(e)$ is the vector $(P, P, \dots, P)$ for some $P \in \mathbf{P}$, then $e \in E_k(\mathbf{P})$ and $g(e) = (g \mid \mathbf{P})(e)$.

Example. Let g be a 3-structure, let $dom(g) = \{x_1, x_2, \dots, x_6\}$, and $P_1 = \{x_1, x_2, x_3\}$, $P_2 = \{x_4\}$, $P_3 = \{x_5, x_6\}$. Suppose $\mathbf{P} = \{P_1, P_2, P_3\}$ is a congruence partition and that the quotient $g/\mathbf{P}$ and the corresponding factors are given. The color in g of $e = (x_1, x_4, x_2)$ is given by the color of $\phi_{\mathbf{P}}(\mathrm{e}) = (P_1, P_2, P_1)$ in $g/\mathbf{P}$. Let $e' = (x_1, x_2, x_2)$. $\phi_{\mathbf{P}}(\mathrm{e}') = (P_1, P_1, P_1) \notin E_3(\mathbf{P})$, so the its color is not given by the quotient. However, e' is an edge in the factor induced by P_1, so its color is given by its color there.

Definition. Let g be a k-structure, and let $X \subseteq dom(g)$. An edge $e \in E_k(dom(g))$ *transcends* X if $supp(e) \cap X \neq \emptyset$ and $supp(e) - X \neq \emptyset$. For a subset $X \subseteq dom(g)$, let $\sim_X$ denote the following equivalence relation. For any pair $e, f \in E_k(X)$, $e \sim_X f$ iff $\pi_i(e) \notin X$ or $\pi_i(f) \notin X$ implies $\pi_i(e) = \pi_i(f)$. X is a *clan* iff for any $e \in E_k(dom(g))$ such that $e \notin E_k(X)$, $e \sim_X f$ implies that $g(e) = g(f)$.

In other words, if there is a pair e, f of edges of g that transcend X, e and f are different colors, and f can be obtained from e by substituting new elements of X for the current elements of X in e, then X is not a clan. If there is no such pair of edges, then X is a clan.

Example. Let g be a 3-structure, let $dom(g)=\{x_1,x_2,x_3,x_4\}$, and let $X=\{x_1,x_2\}$. The edge $e=(x_1,x_3,x_1)$ transcends X, since it contains $x_1 \in X$ and $x_3 \notin X$. The edge $f=(x_2,x_3,x_1) \sim_X e$. If X is a clan, then e and f must be the same color.

Lemma 3.1. *$dom(g)$, the singleton subsets of $dom(g)$, and the empty set are clans of g.*

Proof. $dom(g)$ and the empty set are clans, since no edges transcend them. The singleton sets are clans, since if X is a singleton set, there is no pair of $e,f \in E_k(dom(g))$ such that $e \sim_X f$ and $e \neq f$. □

Definition. If g is a k-structure, then $dom(g)$, the singleton subsets of $dom(g)$, and the empty set are the *trivial* clans of g.

Lemma 3.2. *Let g be a k-structure and let $\mathbf{P}$ be a partition of $dom(g)$. $\mathbf{P}$ is a congruence partition if and only if each member of $\mathbf{P}$ is a clan of g.*

Proof. Follows immediately from the definition of congruence partitions and the definition of a clan. □

Theorem 3.3. *The family of nonempty clans of a k-structure is a point-partitive hypergraph.*

Proof. Let g be a k-structure, and let $\mathbf{F}$ denote its clans. By Lemma 3.1, $dom(g)$ is a clan, so $\bigcup \mathbf{F}=dom(g)$ is a member of $\mathbf{F}$. Also by Lemma 3.1, the singleton subsets of $\bigcup \mathbf{F}$ are members of $\mathbf{F}$. It remains to show that if X and Y are overlapping clans, $X \cup Y, X \cap Y, X-Y$, and $Y-X$ are also clans.

To show that $X \cup Y$ is a clan, let $a \in (X \cap Y)$. Let $e=(x_1,x_2,\ldots,x_k)$ be an edge that transcends $X \cup Y$, and let x_i be a member of e contained in $X \cup Y$. If x_i is in X, then e transcends X, and $e'=(x_1,x_2,\ldots,x_{i-1},a,x_{i+1},\ldots,x_k)$ is the same color as e. An identical argument shows that e' is the same color as e if x_i is in Y. Clearly, e' transcends both X and Y. Since a is in both X and Y, any member of X or Y may be substituted for a to yield an edge of the same color as e. It follows that $X \cup Y$ is a clan of g.

To show that $X \cap Y$ is a clan, let $e=(x_1,x_2,,\ldots,x_k)$ be an edge that transcends $X \cap Y$, and let x_i be a member of e that is in $X \cap Y$. Clearly, e transcends either X or Y. If it transcends Y, then let $y \in Y$. Then $e'=(x_1,x_2,\ldots,x_{i-1},y,x_{i+1},\ldots,x_k)$ is the same color as e is. This is true if y is any member of $X \cap Y$, so $X \cap Y$ is a clan of g. An identical argument shows the same result if e transcends X.

To show that $X-Y$ is a clan, let $e=(x_1,x_2,\ldots,x_k)$ be an edge that transcends $X-Y$. Then e contains an element x_i that is in $X-Y$. Let a be any member of $X-Y$.

It suffices to show that $f=(x_1,x_2,\dots,x_{i-1},a,x_{i+1},,\dots,x_k)$ is the same color as e. If e transcends X, this follows from the fact that X is a clan. Suppose e does not transcend X. Then e contains an element $x_j \in X \cap Y$. Let $b \in Y-X$. Let $e'=(x_1,x_2,\dots,x_{j-1},b,x_{j+1},\dots,x_k)$. Since Y is a clan, e' is the same color as e. However, e' transcends X. Thus, substituting a for x_i in e' gives another edge e'' of the same color. Since e'' transcends Y, substituting x_j for y in e'' gives another edge of the same color. However, this last edge is f, giving the result.

By symmetry, $Y-X$ is a clan. $\square$

Definition. A clan X of g is *prime* if it overlaps no other clan of g. The *prime tree family* of g is the family of nonempty prime clans of g. If X is a nonsingleton member of the prime tree family, $children_g(X)$ denotes the maximal-cardinality members of the prime tree family that are proper subsets of X.

By Theorems 3.3 and 2.1, the prime tree family is a representation for the family of clans of a k-structure. This representation requires $O(n)$ space, where n is the number of nodes in the k-structure. By Lemma 3.2, this is also a representation of all congruence partitions on the k-structure.

4. Properties of quotients and factors

Definition. Given two k-structures g and g', suppose there exists a mapping ϕ from $dom(g)$ to $dom(g')$ such that ϕ is one-to-one and onto, and such that whenever e is an edge of g, e and $\phi(e)$ are the same color. Then g' is a *renaming* of g.

Proposition 4.1. *Let g be a k-structure, let* $\mathbf{P}$ *be a congruence partition on g, and let S be a system of representatives from* $\mathbf{P}$. *Then $g \mid S$ is a renaming of $g/\mathbf{P}$.*

Thus, we may use $g \mid S$ and $g/\mathbf{P}$ interchangeably. The image in $g/\mathbf{P}$ of a node x of g is the member of $\mathbf{P}$ that contains x. When $g \mid S$ is viewed as the quotient, the image of x is the representative in S of that member of $\mathbf{P}$.

Proposition 4.2 (Restriction rule). *Let g be a k-structure, let X be a clan of g, and let Y be a subset of $dom(g)$. Then $X \cap Y$ is a clan of $g \mid Y$.*

Proposition 4.3 (Autonomous substructure rule). *Let g be a k-structure, and let Y be a clan of g. The clans of $g \mid Y$ are those clans of g that are contained in Y.*

Theorem 4.4 (Quotient rule). *Let g be a k-structure, and let* $\mathbf{P}$ *be a family of clans of g that partitions $dom(g)$. Then*

(1) *If X is a clan of g, the image of X in $g/\mathbf{P}$ is a clan in $g/\mathbf{P}$;*
(2) *If Y is a clan of $g/\mathbf{P}$ then its inverse image is a clan in g.*

Proof. (1) Let X be a clan of g. Let $\mathbf{P}_1=\{P\cap X: P\in\mathbf{P}$ and $P\cap X\neq\emptyset\}$. Let $\mathbf{P}_2=\{P: P\in\mathbf{P}$ and $P\cap X$ is empty$\}$. Let S_1 be a system of representatives from $\mathbf{P}_1$, let S_2 be a system of representatives from $\mathbf{P}_2$, and let $Y=S_1\cup S_2$. By the restriction rule (4.2), $S_1=X\cap Y$ is a clan of $g\,|\,Y$, since X is a clan of g. Y is a system of representatives from $\mathbf{P}$, so $g\,|\,Y$ is a renaming of $g/\mathbf{P}$. S_1 is a renaming of the image of X in $g/\mathbf{P}$, and since S_1 is a clan of $g\,|\,Y$, we conclude that the image of X in $g/\mathbf{P}$ is a clan in $g/\mathbf{P}$.

(2) Let X' be a clan of $g/\mathbf{P}$, and let X be its inverse image in g. Suppose X is not a clan of g. Then there exist two distinct edges e_1 and e_2 of g such that $e_1\sim_X e_2$, but $g(e_1)\neq(e_2)$. Let $\mathbf{e}'_1=\phi_\mathbf{P}(e_1)$ and let $\mathbf{e}'_2=\phi_\mathbf{P}(e_2)$. That is $\mathbf{e}'_1$ and $\mathbf{e}'_2$ are the images of e_1 and e_2, respectively, in $g/\mathbf{P}$. Since $g(e_1)=(g/\mathbf{P})(\mathbf{e}'_1)$ and $g(e_2)=(g/\mathbf{P})(\mathbf{e}'_2)$, it follows that $(g/\mathbf{P})(\mathbf{e}'_1)\neq(g/\mathbf{P})(\mathbf{e}'_2)$. Since $(g/\mathbf{P})(\mathbf{e}'_1)\sim_{X'}(g/\mathbf{P})(\mathbf{e}'_2)$, this contradicts the assumption that X' is a clan in $g/\mathbf{P}$. □

The restriction rule (4.2), the autonomous substructure rule (4.3), the quotient rule (4.4) and Theorem 3.3 are the fundamental properties of the decomposition, and are the basis of the subsequent results of this paper.

Definition. Let g be a k-structure. Then g is *primitive* if all of its clans are trivial. It is *complete* if every subset of $dom(g)$ is a clan of g. It is *linear* if there exists a linear ordering of $dom(g)$ such that a $X\subseteq dom(g)$ is a clan if and only if the members of X are consecutive in the linear order.

If g is a k-structure and $|dom(g)|<3$, g is clearly primitive, complete, and linear. If $dom(g)\geqslant 3$, it can fall into at most one of these categories. The following definition is useful.

Definition. If g is a k-structure, then g is *strongly primitive*, *strongly complete*, or *strongly linear* if $|dom(g)|\geqslant 3$ and it is primitive, complete, or linear, respectively.

Lemma 4.5. *If all edges of a k-structure are the same color, it is complete. If a k-structure is strongly complete, all of its edges are the same color.*

Proof. Let g be a k-structure. If all edges of g are the same color, it is clearly complete. Conversely, suppose g is strongly complete. Thus, it has at least three nodes. Let e be an arbitrary edge of g. Let a *substitution operation* on e denote the result of changing e to a different edge e' by changing the node in position i of e to a different node of g, subject to the constraint that $|supp(e)\cup supp(e')|\geqslant 3$. Let x be the old node and let y be the new node. Since g is complete, $\{x,y\}$ is a clan of g. Since $|supp(e)\cup supp(e')|\geqslant 3$, e and e' transcend $\{x,y\}$. Thus, e and e' are the same color. Any edge of g may be obtained from any other by a series of substitution operations, so all of edges of g are the same color. □

Lemma 4.6. *Let g be a k-structure, and let X be a nonsingleton prime clan of g. Then $(g\,|\,X)/children_g(X)$ is primitive, complete, or linear. $(g\,|\,X)/children_g(X)$ is primitive, complete, or linear if and only if X is* q-*primitive,* q-*complete, or* q-*linear, respectively.*

Proof. Follows from Theorems 2.1 and 3.3, and the quotient rule (4.4) $\square$

The terms q-primitive, q-linear, and q-complete are derived from the fact that the *quotient* $(g\,|X)/children_g(X)$ is primitive, linear, or complete. *Strongly q-primitive* means that this quotient is strongly primitive. *Strongly q-linear* and *strongly q-complete* are defined similarly.

Theorem 4.7. *There exists no strongly linear k-structure when $k>2$.*

Proof. Suppose there is a linear arrangement of nodes of a k-structure so that each set of consecutive elements is a clan. Let $(1,2,3,\ldots,n)$ denote the nodes in this order. We use the property that if X is a clan and $e=(x_1,\ldots,x_k)$ is an edge that transcends X, then the elements of X may be freely substituted in e for the elements $x_i \in X$ without changing the color of the resulting edge. Any edge of the k-structure may be obtained from any other by a series of such substitutions.

Consider $e=(1,n,2,2,\ldots,2)$, and suppose it is red. Any edge with a single 1, a single n, and the rest 2's are also red, which can be seen as follows. To move the n to a position occupied by a 2, change that 2 to a n and change the original n to a 2. The result is a red edge since $\{2,\ldots,n\}$ is a clan, and the edge at each transition contains 1 and a member of $\{2,\ldots,n\}$. Moving a 1 to a position occupied by a 2 is done the same way, using the fact that $\{1,2\}$ is a clan. Moving the 1 or n to arbitrary positions can be done with one or more moves where you move the 1 or n to a position occupied by a 2.

Consider an arbitrary edge $f=(x_1,x_2,\ldots,x_k)$. Let x_i be a minimum element of f, and let x_j be a maximum one. Note that $x_i<x_j$, since f is a heterogeneous sequence. Change x_i to a 1. This does not change the color, since $\{1,2,\ldots,x_{j-1}\}$ is a clan that contains x_i, and not x_j. Change x_j to n. This does not change the color, since $\{2,3,\ldots,n\}$ is a clan that contains x_j, but not $x_i=1$. Similarly, the other entries can all be changed to 2's, either because $\{1,2,\ldots,n-1\}$ is a clan or because $\{2,3,\ldots,n\}$ is a clan. The result is an edge with one 1, one n, and the rest 2's, which is red. Thus, every edge of the k-structure is red, which means it is complete, not linear, by Lemma 4.5. $\square$

Corollary. *If g is a k-structure where $k>2$, every member of its prime tree family is q-primitive or q-complete.*

5. A hereditary property of primitive k-structures

Two k-structures g and h are *isomorphic* if there exists a one-to-one mapping ϕ of the nodes of g onto the nodes of h such that e and e' are the same color in g if and only if $\phi(e)$ and $\phi(e')$ are the same color in h. This does not require that e and $\phi(e)$ be the same color.

The prime tree decomposition is important partly because it shows how a nonprimitive k-structure may be described in terms of smaller k-structures. However, the decomposition gives nothing meaningful when the k-structure is primitive. When a k-structure is nonprimitive, the decomposition gives the k-structure in terms of smaller k-structures that are primitive, complete, or linear. There is only one complete k-structure on n nodes for any k, and only one linear 2-structure on n nodes, up to isomorphism. Their structure is trivially given in terms of smaller substructures. On the other hand, even in the restricted case of partial order graphs, the proportion of graphs that are primitive goes to 1.0 as n goes to infinity [21]. Thus, complete and linear k-structures are special cases, while the primitive class is a quite general category for the remaining k-structures in the decomposition.

Of fundamental importance, therefore, is how primitive k-structures may be described in terms of smaller structures. Ehrenfeucht and Rozenberg [11] and Schmerl and Trotter [24] give the following theorem about primitive 2-structures.

Theorem 5.1. *Let g be a strongly primitive 2-structure. There exists a set $D' \subset dom(g)$ such that $g|D'$ is primitive, and $|D'| \geq |dom(g)| - 2$.*

This theorem shows that strongly primitive 2-structures may be expressed recursively in terms of a chain of ever-smaller strongly primitive substructures, the smallest having either three or four nodes. (This statement of the theorem differs from the original, since we include two-node 2-structures in the class of primitive 2-structures.)

Consider the primitive k-structure where an edge e is green if $supp(e)=k$, and red if $supp(e)<k$. The substructure induced by k or more nodes is primitive, but the substructure induced by fewer than k nodes has only red edges, and is thus complete. It follows that any primitive substructure on k nodes contains no primitive substructure on greater than two nodes. Thus, Theorem 5.1 does not hold for k-structures. In addition, the techniques of [11] do not lead to an obvious generalization of Theorem 5.1 to k-structures.

The main result of this section is the following generalization of Theorem 5.1

Theorem 5.2. *Let g be a strongly primitive k-structure. There exists a set $D' \subset dom(g)$ such that $g|D'$ is primitive, and $|D'| \geq |dom(g)| - k$.*

This theorem shows that a primitive k-structure may be expressed in terms of a chain of ever-smaller primitive substructures. However, the difference in size between consecutive substructures in the chain is at most k instead of at most 2.

Definition. Let g be a k-structure. Let g_{-x} denote $g|(dom(g)-\{x\})$, and let $g_{-x,y}$ denote $g|(D'-\{x,y\})$. A set $X \subseteq dom(g)$ is a *quasiclan* if there exists $x \in dom(g)$ such that $X-\{x\}$ is a clan in g_{-x}. Any subset of the set $\{z: z\in dom(g)$ and $X-\{z\}$ is a clan in $g_{-z}\}$ is an *enabling set* for X.

Note that $dom(g)$ and its singleton subsets are trivial quasiclans.

Lemma 5.3. *Let g be a k-structure, and suppose that X is a quasiclan of g, Y is an enabling set for X, $|Y|>k$, and $X-Y$ is nonempty. Then X is a clan of g.*

Proof. Let $e=(x_1,x_2,\dots,x_k)$ be any edge of g that transcends X. Find i such that $x_i\in X$. Let e' be an edge obtained from e by making x_i into a different node of X. It suffices to show that e and e' must always be the same color. To do this, we let e'' be an edge obtained from e by making x_i be a node of $X-Y$. Then $|(supp(e)\cup supp(e''))\cap Y|\leqslant k$. Since $|Y|>k$, there exists $y\in Y$ such that e and e'' are edges in g_{-y}. $X-\{y\}$ is a clan in g_{-y} and both e and e'' transcend $X-\{y\}$, so e and e'' are the same color. Identical reasoning shows that e' and e'' are the same color. Transitively, e and e' are the same color. □

The thrust of the proof of Theorem 5.2 is the following. If g is a k-structure with a strongly primitive substructure, let D' be a maximum-cardinality proper subset of $dom(g)$ such that $g\,|\,D'$ is primitive. We define a type of nontrivial subset of $dom(g)$ called a *sprout*. If a sprout exists, we show that it is a quasiclan, it contains one element of D', and $dom(g)-D'$ is an enabling set. If $|dom(g)-D'|>k$, then by Lemma 5.3, the sprout is a clan of g. Similarly, if no sprout exists, then we show that D' is a quasiclan and $dom(g)-D'$ is an enabling set, so if $|dom(g)-D'|>k, D'$ is a clan of g. Thus, if $|dom(g)-D'|>k$, g is not primitive. This establishes Theorem 5.2 for any primitive k-structure that has a strongly primitive substructure.

The foregoing is established in Lemma 5.8 below. To complete the proof of Theorem 5.2, we then show that any strongly primitive k-structure with more than $k+2$ nodes has a strongly primitive substructure. This is obtained by a similar applications of quasiclan idea of Lemma 5.3 and the fundamental properties of the prime tree decomposition given by the restriction rule (4.2), the autonomous substructure rule (4.3), the quotient rule (4.4), and Theorem 3.3.

Lemma 5.4. *Let g be a k-structure such that $|dom(g)|>2$, and let D' be a maximum-cardinality proper subset of $dom(g)$* such that *$g\,|\,D'$ is primitive. Then for any $x\in dom(g)-D'$ and any nontrivial clan X of $g\,|\,D'\cup\{x\})$, either $X=D'$, or $X=\{x,x'\}$, where $x'\in D'$.*

Proof. Any one- or two-element subset of $dom(g)$ induces a primitive substructure in g, so D' exists. The lemma is obvious when $|D'|=2$. If $|D'|>2$ and a nontrivial clan X fails to satisfy the lemma, then $X\cap D'$ is a nontrivial clan of $g\,|\,D'$ by the restriction rule (4.2). This contradicts the assumed primitivity of $g\,|\,D'$. □

Definition. Let g be a k-structure, let $D' \subseteq dom(g)$ be a maximal-cardinality subset of $dom(g)$ such that $g \mid D'$ is primitive, and let $x \in dom(g) - D'$. If D' is a clan in $g \mid (D' \cup \{x\})$, then x is *global for D in g*. If $\{x, x'\}$ is a clan for some $x' \in D'$, then x is *local for D' in g*. If x is both global and local for D', then x is *mixed for D' in g*.

Lemma 5.5. *Let g be a nonprimitive k-structure that has a strongly primitive substructure. Let D' be a maximum-cardinality subset of dom(g) such that g | D' is strongly primitive. If x∈dom(g)−D', then x is either local or global for D', but not mixed. If it is local, then there is a unique x'∈D' such that {x, x'} is a clan of g | (D'∪{x}).*

Proof [11]. By Lemma 5.4, x is global or local for D'.

$|D'| > 2$, since $g \mid D'$ is strongly primitive. Suppose x is both global and local for D'. Then $\{x, y\}$ and D' are overlapping clans of $g \mid (D' \cup \{x\})$, for some $y \in D'$. $D' - \{y\}$ is a clan of $g \mid (D' \cup \{x\})$ by Theorem 3.3. By the restriction rule (4.2), it is a nontrivial clan of $g \mid D'$, a contradiction. Suppose $\{x, y\}$ and $\{x, z\}$ are two clans of $g \mid (D' \cup \{x\})$ for some $z \in D'$ and $z \neq y$. The clans overlap, so $\{x, y, z\}$ is also a clan of $g \mid (D' \cup \{x\})$. By the restriction rule (4.2), $\{y, z\}$ is a nontrivial clan of $g \mid D'$, a contradiction. Thus, $x' = y$ is unique. □

Definition. Let g, D', and x be as in Lemma 5.5. If x is local, then $uni_g(D', x)$ denotes the unique $x \in D'$ such that $\{x, x'\}$ is a clan of $g \mid (D' \cup \{x\})$. The set $\{x'\} \cup \{y: uni_g(D', y) = x'\}$ is a *sprout* on D' in g, and x' is the sprout's *stem*.

Lemma 5.6. *Let g be a nonprimitive k-structure that has a strongly primitive substructure, and let D' be a maximum-cardinality subset of its domain such that g | D' is strongly primitive. Any nontrivial clan X of g has one of the following three properties*

(1) *It contains D' and all of its sprouts.*
(2) *It is contained in a sprout on D'.*
(3) *It is disjoint from D' and its sprouts.*

Proof. $D' \subseteq X$ or $|X \cap D'| \leqslant 1$, since $X \cap D'$ is a clan of $g \mid D'$ by the restriction rule (4.2), and $g \mid D'$ is primitive.

Suppose $D' \subseteq X$, and there is an element x that is local for D' but not contained in X. By the restriction rule (4.2), $X \cap (D' \cup \{x\}) = D'$ is a clan of $g \mid (D' \cup \{x\})$. Thus, x is global. By Lemma 5.5, x cannot be local, a contradiction. It follows that X contains D' and all of its local elements, hence all of its sprouts.

Suppose $|X \cap D'| = 1$. Let x' be the element of $|X \cap D'|$, and let x be any other element of X. By the restriction rule (4.2), $X \cap (D' \cup \{x\}) = \{x, x'\}$ is a clan of $D' \cup \{x\}$, so $uni(D', x) = x'$ for any $x \in X - \{x'\}$. Thus, X is contained in a sprout.

Suppose $X \cap D' = \emptyset$. Let $x \in X$. X is a clan in $g \mid (D' \cup X)$ by the restriction rule (4.2). $h = g \mid (D' \cup \{x\})$ gives a quotient of this substructure, where the image of X is x, and the image of each node of D' is itself. If x is local for D', then $\{x, uni(D', x)\}$ is a clan in h, and its inverse image, $X \cup \{uni(D', x)\}$ is a clan of g by the quotient rule (4.4). This clan has one element in common with D', so it is contained in a sprout by the result of the previous paragraph. Otherwise, if x is global for D', then D' is a clan in h by the restriction rule (4.2). Its inverse image, D', is a clan of $g|(D' \cup X)$. By the restriction rule (4.2), D' is a clan in $g|(D' \cup \{y\})$ for any $y \in X$, so X consists of global elements, and is disjoint from any sprout. □

A version of Lemma 5.6 for graphs is given as Theorem 1 of [5],

Lemma 5.7. *Let g be a nonprimitive k-structure that has a strongly primitive substructure, and let D' be a maximum-cardinality subset of $dom(g)$ that induces a strongly primitive substructure. Each sprout on D' is a clan in g, and the union of D' and its sprout is also a clan in g.*

Proof. Let Z be the smallest clan of g that contains D'. Z is unique, since if there were two such clans, their intersection would be a smaller one still, by Theorem 3.3 Let $g' = g|Z$. Since Z is a clan of g, any clan of g' is a clan of g by the autonomous substructure rule (4.3).

Suppose there is a nontrivial clan X of g'. X cannot contain D', since if it did, X would be a clan of g that contradicted the definition of Z. By Lemma 5.6, either $|X \cap D'| = 0$ or $|X \cap D'| = 1$. If $|X \cap D'| = 1$, let x' be the element of $X \cap D'$. If $|X \cap D'| = 0$, let x' be an arbitrary element of X. Let $h = g'|((dom(g') - X) \cup \{x'\})$. h is a quotient of g', where X is mapped to x' and each element of $dom(g') - X$ is mapped to itself.

No nontrivial clan of h can contain D', since the inverse image of such a clan would be a nontrivial clan of g' that contained D', which we have shown cannot occur. Thus, if $|dom(h)| > |D'|$, h has a nontrivial clan that falls into Lemma 5.6 (2) or (3). We can thus apply the reasoning of the previous paragraph recursively on h to get a series of successive quotients. Eventually, this yields a quotient whose domain is D'. Each quotient arising from a clan that falls into Lemma 5.6(2) reduces the number of local elements in the result, and each quotient arising from one from Lemma 5.6(3) reduces the number of global elements. However, a clan from Lemma 5.6(3) could never remove the last global element, so we can conclude that Z has no global elements, and that all quotients in the series arise from clans falling into Lemma 5.6(2). By Lemma 5.6(1) Z is the union of D' and its sprouts.

In the composition of the series of quotient mappings, each sprout of g' maps to its stem. Since the stem is a one-element set, hence a clan, each sprout is a clan of g' by the quotient rule (4.4). Since the clans of g' are clans of g, each sprout is a clan of g. □

Lemma 5.8. *Let g be a primitive k-structure that has a strongly primitive proper substructure. Let D' be a maximum-cardinality proper subset of $dom(g)$ such that $g|D'$ is primitive. Then $|dom(g) - D'| \leq k$.*

Proof. Suppose the lemma does not hold. Then $|dom(g)-D'|\geqslant k+1$. Let $Y=dom(g)-D'$. D' and the stem of each sprout on D' in g are not in Y. For any $y\in Y$, g_{-y} is not primitive, and D' is a maximum-cardinality subset of g_{-y} that induces a primitive substructure. If there are elements of $dom(g)$ that are local for D', let S be a sprout. $S-\{y\}$ is a singleton set or a sprout in g_{-y}, hence a clan in g_{-y} by Lemma 5.7. S is a quasiclan in g. Y is an enabling set for S, and since $|Y|\geqslant k+1$, S is a clan of g by Lemma 5.3. If there are no elements of Y that are local for D', then D' is a clan of g_{-y} by Lemma 5.7. Y is an enabling set for D', so D' is a nontrivial clan of g by Lemma 5.3. In either case, g is not primitive. □

Lemma 5.8. gives Theorem 5.2 for all primitive substructures with strongly primitive proper substructures. It remains to show that if g is a primitive k-structure with at least $k+3$ nodes, it has a strongly primitive substructure. This is the subject of the remainder of this section.

Lemma 5.9. *Let g be a nonprimitive k-structure such that $|dom(g)|\geqslant 3$. If g has no strongly primitive substructure, then g has a doubleton clan.*

Proof. Let X be a smallest nontrivial clan of g. If $|X|>2$, then X has no proper subset that is a nontrivial clan of g. By the autonomous substructure rule (4.3), $g\,|\,X$ contains no nontrivial clan, and is a strongly primitive substructure of g, a contradiction. □

Lemma 5.10. *Let g be a nonprimitive k-structure that has no strongly primitive substructure. A doubleton subset D' of $dom(g)$ is a clan in g if and only if every element of $dom(g)-D'$ is global or mixed for D'.*

Proof. (only if) Suppose D' is a clan of g. By the restriction rule (4.2), it is a clan of $g\,|(D'\cup\{x\})$ for any $x\in dom(g)-D'$. Thus, x is global or mixed for D'.

(if) Let D' be an arbitrary doubleton set such that every element of $dom(g)-D'$ is global or mixed for D'. Let W be the smallest clan of g that contains D', and let $h=g\,|\,W$. W is unique, since the intersection of two such clans would be a still smaller clan containing D' by Theorem 3.3. Then h has no nontrivial clan that contains D' by the restriction rule (4.2).

Suppose $W=dom(h)\neq D'$. If $|dom(h)\geqslant 4$, h has a doubleton clan, X, by Lemma 5.9. There exists $x\in X-D'$, since D' is not a clan of h, hence $D'\neq X$. $h'=h\,|(dom(h)-\{x\})$ is a quotient of h, where X maps to the element of $X-\{x\}$ and the remaining elements of $dom(h)$ map to themselves. If D' is a clan in h', the inverse image of D' is a nontrivial clan in h that contains D', a contradiction. Since D' is not a clan in h', we may apply the above argument recursively until we eventually derive a substructure h'' on three elements that contains D' and such that D' is not a clan in h''. If $\{y\}=dom(h'')-D'$, y is neither global nor mixed for D', a contradiction. Our assumption that $W\neq D'$ leads to a contradiction, so $W=D'$, proving that D' is a clan. □

Lemma 5.11. *Let g be a k-structure on $n \geqslant 3$ nodes that has no strongly primitive proper substructure, and let x and y be distinct elements of $dom(g)$. If there is a set X that is a nontrivial clan in both g_{-x} and g_{-y}, then g is not primitive.*

Proof. Since $g \mid X$ contains no strongly primitive substructure, it has a doubleton clan, D', by Lemma 5.9. Since X is a clan in both g_{-x} and g_{-y}, D' is a clan in both g_{-x} and g_{-y} by the autonomous substructure rule (4.3). Since D' is a clan of g_{-x}, all elements of $dom(g)-D'-\{x\}$ are global or mixed for D', by Lemma 5.10. By identical reasoning, all elements of $dom(g)-D'-\{y\}$ are global or mixed for D'. It follows that all elements of $dom(g)-D'$ are global or mixed for D', so D' is a clan of g_{-z} for any $z \in dom(g)-D'$, by Lemma 5.10. By Lemma 5.3, D' is a clan of g. □

Lemma 5.12. *Let g be a nonprimitive k-structure that has no strongly primitive substructure. Then the prime tree family of g has no strongly q-primitive clans.*

Proof. A system of distinct representatives from the children of a strongly q-primitive member of the prime tree family would induce a strongly primitive substructure in g, a contradiction. □

Lemma 5.13. *Let g be a nonprimitive k-structure, where $k>2$, and suppose g has no strongly primitive substructure. Let $X \subseteq dom(g)$ such that $|X| \geqslant 2$. Then there are two disjoint clans of g, Y_l and Y_2, such that $Y_1 \cap X$ and $Y_2 \cap X$ are a nontrivial partition of X.*

Proof. Let Z be the least common ancestor of the members of X in the prime tree family of g. By Lemma 5.12 and the corollary to Lemma 4.6, Z is q-complete. Let U be one of Z's children that contains a member of X. U and $Z-U$ are disjoint clans of g that satisfy the definition of Y_1 and Y_2. □

Lemma 5.14. *Let X be a clan of g_{-x} and suppose $y \in dom(g_{-x})-X$. The intersection of X and any clan of g_{-y} is a clan in g_{-x}.*

Proof. Let Y be a clan of g_{-y}. $Y-\{x\}$, and $X-\{y\}$ are clans of $g_{-x,y}$ by the restriction rule (4.2). $(X-\{y\}) \cap (Y-\{x\}) = X \cap Y$ is a clan of $g_{-x,y}$ by Theorem 3.3. $X \cap Y$ is then a clan of $g \mid X$ by the restriction rule (4.2). Since X is a clan of g_{-x}, $X \cap Y$ is a clan of g_{-x} by the autonomous substructure rule (4.3). □

Lemma 5.15. *Let g be a k-structure, let X be a nontrivial clan of g_{-x}, let $y \in dom(g_{-x})-X$, and let Y be a clan of g_{-y} such that $X \cap Y$ is nonempty.*

(1) *If Y does not contain x, then $X \cap Y$ is a clan in both g_{-x} and g_{-y}*

(2) *If Y contains x and $|Y-X| \geqslant 2$, then $X-Y$ is a clan in both g_{-x} and g_{-y}.*

Proof. (1) Applying Lemma 5.14 to both X and Y gives (1).

(2) If $X-Y$ is empty, it is trivially a clan in both g_{-x} and g_{-y}. Otherwise, X and $Y-\{x\}$ are overlapping clans of $g_{-x,y}$. Thus $X-(Y-\{x\})=X-Y$ is a clan of $g_{-x,y}$. By the restriction rule (4.2), $X-Y$ is a clan of $g|X$, and by the autonomous substructures rule (4.3), $X-Y$ is a clan of g_{-x}.

It remains to show that $X-Y$ is a clan of g_{-y}. Let $U=dom(g_{-y})-Y)\cup\{x'\}$ for some $x'\in X\cap Y$. U does not contain x, so $U\subseteq dom(g_{-x})$. $(X-Y)\cap U=X-Y$ is a clan of $g|U$ by the restriction rule (4.2) and the fact that $X-Y$ is a clan of g_{-x}. However, $g|U$ is the quotient of g_{-y} obtained by mapping Y to x' and mapping each remaining element of $dom(g_{-y})$ to itself. Since $X-Y$ is a clan in this quotient, its inverse image, $X-Y$, is also a clan in g_{-y} by the quotient rule (4.4). $\square$

Lemma 5.16. *If g is a primitive k-structure with at least $k+3$ nodes, then g has a strongly primitive substructure.*

Proof. The lemma has been proven for the case of $k=2$ [11]. Suppose that $k\geqslant 3$, that g is a primitive k-structure on at least $k+3$ nodes, and that g has no strongly primitive proper substructure. We will prove the lemma by showing that these assumptions lead to a contradiction.

Claim 1. *There is no $W\subset dom(g)$ and $u,v\in dom(g)$ such that W is a nontrivial clan of both g_{-u} and g_{-v}.*

Proof of Claim 1. If such a W exists, then g is nonprimitive by Lemma 5.11, a contradiction.

Claim 2. *Let x be a member of $dom(g)$. If X is a nontrivial clan in g_{-x}, and $y\in dom(g_{-x})-X$, there is a clan Y of $dom(g_{-y})$ such that $|X\cap Y|=|X|-1$.*

Proof of Claim 2. X is a subset of $dom(g_{-y})$. By Lemma 5.13, there are two disjoint clans, Y_1 and Y_2, of g_{-y}, such that $X\cap Y_1$ and $X\cap Y_2$ is a nontrivial partition of X. At most one of $\{Y_1, Y_2\}$ may contain x; without loss of generality, assume Y_1 does not contain x. $Y_1\cap X$ is a clan of both g_{-x} and g_{-y}, by Lemma 5.15 (1). By Claim 1, $|Y_1\cap X|=1$. Thus, $|Y_2\cap X|=|X|-1$.

Claim 3. *Let x be a member of $dom(g)$. If X is a nontrivial clan of g_{-x}, then X contains a clan of g_{-x} that has $|X|-1$ elements.*

Proof of Claim 3. Follows from Claim 2 and Lemma 5.14.

Claim 4. *For any $x\in dom(g)$, if X is a nontrivial clan in g_{-x} such that $|X|\geqslant 3$, and $y\in dom(g_{-x})-X$, then there is a clan Y of g_{-x} that consists of x and $|X|-1$ elements of X.*

Proof of Claim 4. By Claim 2, there is a clan Y' of g_{-y} that contains $|X|-1$ elements of X. If Y' does not contain x, then by Lemma 5.15 (1), $Y' \cap X$ is a nontrivial clan of g_{-x} and g_{-y}, a contradiction by Claim 1. Y' contains x and $|X|-1$ elements of X.

If $Y'-(X \cup \{x\})$ is empty, Claim 4 is satisfied. Otherwise, Y' contains elements of $dom(g)-(X \cup \{x\})$. By Claim 3, Y' contains a clan of g_{-y}, Y'', such that $|Y'-Y''|=1$. Y'' must contain x for the same reason that Y'' must contain x. If $Y'-Y''$ consists of a member of X, then $X-Y''$ is a nontrivial clan of both g_{-x} and g_{-y} by Lemma 5.15 (2), a contradiction. Thus, $Y'-Y''$ consists of an element of $dom(g)-(X \cup \{x\})$. Applying this argument recursively to Y'' eventually gives a clan of g_{-y} that satisfies the claim.

To prove Lemma 5.16, we observe that by Lemma 5.13, there is a partition of $dom(g_{-x})$ into two clans of g_{-x}. Let X' be the larger of the two. Since $k \geqslant 3$, $|dom(g)| \geqslant 6$, and $dom(g_{-x})$ has at least five elements. Thus, $|X'| \geqslant 3$. By Claim 3, there is a clan X of g_{-x} such that $|X|=3$. Let $y, z \in dom(g_{-x})-X$. By Claim 4, there is a clan Y in g_{-y} and a clan Z in g_{-z} that each consist of x and two elements of X. $|Y \cap Z| \geqslant 2$, Y does not contain z, and Z does not contain y. By Lemma 5.15 (1), $Y \cap Z$ is a nontrivial clan of both g_{-y} and g_{-z}, contradicting Claim 1. □

Theorem 5.2 now follows immediately from Lemmas 5.8 and 5.16, giving the main result of this section.

6. Future work

$O(n+m\alpha(m,n))$ and $O(n+m)$ algorithms [27, 18] have recently been developed for computing the prime tree family of an undirected graph, and $O(n^2)$, algorithms [7, 17, 23] are known for computing the prime tree family for graphs and 2-structures. Möhring [20] gives a proof that the decomposition for k-ary relations may be solved within a time bound that is at least polynomial in n and k, and his proof generalizes easily to k-structures. Other than this result, the efficiency of decomposition algorithms on k-structures is an open question.

The decomposition and hereditary theorems for 2-structures have been generalized to 2-structures on infinite domains [15]. We have restricted our study to k-structures on finite domains, so the generalization of our results to k-structures on infinite domains is still an open question.

Theorem 5.1 shows that every strongly primitive 2-structure has a primitive substructure on at least $n-2$ nodes. Bonizzoni [3] and Schmerl and Trotter [24] have shown that this is the tightest possible lower bound, by giving examples of primitive 2-structure such that no set of $n-1$ nodes induces a primitive substructure. We have shown that $n-2$ is not a lower bound for k-structures, and our example shows that, in general, the lower bound cannot exceed $n-k+2$. On the other hand, Theorem 5.2 shows that $n-k$ is a lower bound. It is an open problem whether $n-k, n-k+1$, or $n-k+2$ is the tightest possible lower bound.

Acknowledgment

The authors would like to thank Hal Gabow for pointing out the relationship between the prime tree decomposition of 2-structures and the previous work on the modular decomposition of graphs.

References

[1] A. Blass, Graphs with unique maximal clumpings, *J. Graph Theory* **2** (1978) 19–24.
[2] B. Bollabas, *Extremal Graph Theory* (Academic Press, New York, 1978).
[3] P. Bonizzoni, Primitive 2-structures with the $(n-2)$ property, *Theoret. Comput. Sci.* **132** (1994) 151–178.
[4] M. Chein, M. Habib and M.C. Maurer, Partitive hypergraphs, *Discrete Math.* **37** (1981) 35–50.
[5] A. Cournier and M. Habib, An efficient algorithm to recognize prime undirected graphs, Tech. Report R.R. LIRMM no. 92-023, Laboratoire D'Informatique, de Robotique et de Microelectronique de Montpellier, 1992.
[6] W.H. Cunningham and J. Edmonds, A combinatorial decomposition theory, *Canadian J. Math.* **32** (1980) 734–765.
[7] A. Ehrenfeucht, H.N. Gabow, R.M. McConnell, and S.J. Sullivan, An $O(n^2)$ divide-and-conquer algorithm to compute the prime tree decomposition of two-structures and modular decomposition of graphs, *J. Algorithms* **16** (1994) 283–294.
[8] A. Ehrenfeucht, T. Harju and G. Rozenberg, Invariants of 2-structures on groups of labels, 1993, manuscript.
[9] A. Ehrenfeucht and G. Rozenberg, Theory of 2-structures, Part 1: Clans, basic subclasses, and morphisms, *Theoret. Comput. Sci.* **70** (1990) 277–303.
[10] A. Ehrenfeucht and G. Rozenberg, Theory of 2-structures, Part 2: Representations through labeled tree families, *Theoret. Comput. Sci.* **70** (1990) 305–342.
[11] A. Ehrenfeucht and G. Rozenberg, Primitivity is hereditary for 2-structures, *Theoret. Comput. Sci.* **70** (1990) 343–358.
[12] T. Gallai, Transitiv orientierbare graphen, *Acta Math. Acad. Sci. Hungar* **18** (1967) 25–66.
[13] M.C. Golumbic, *Algorithmic Graph Theory and Perfect Graphs* (Academic Press, New York, 1980).
[14] M. Habib and M.C. Maurer, On the X-join decomposition for undirected graphs, *Discrete Appl. Math.* **1** (1979) 201–207.
[15] T. Harju and G. Rozenberg, Decompositions of infinite 2-structures, 1993, manuscript.
[16] D. Kelly, Comparability graphs, in: I. Rival, ed., *Graphs and Orders* (Reidel, Boston, 1985) 3–40.
[17] R.M. McConnell, An $O(n^2)$ incremental algorithm to compute the modular decomposition of a 2-structure, *Algorithmica*, to appear.
[18] R.M. McConnell and J.P. Spinrad, Linear-time modular decomposition and efficient transitive orientation of comparability graphs, *Proceedings of the Fifth Annual ACM–SIAM Symposium on Discrete Algorithms*, Arlington, VA, January 1994, 536–545.
[19] R.H Möhring, Algorithmic aspects of comparability graphs and interval graphs, in: I. Rival, ed., *Graphs and Orders* (Reidel, Boston, MA, 1985) 41–101.
[20] R.H. Möhring, Algorithmic aspects of the substitution decomposition optimization over relations, set systems and boolean functions, *Ann. Oper. Res* **4** (1985/6) 195–225.
[21] R.H. Möhring, Almost all comparability graphs are UPO, *Discrete Math.* **50** (1984) 63–70.
[22] R.H. Möhring and F.J. Radermacher, Substitution decomposition and connections with combinatorial optimization, *Ann. Discrete Math.* **19** (1984) 257–356.
[23] J.H. Muller and J. Spinrad, Incremental modular decomposition, *J. ACM* **36** (1989) 1–19.
[24] J.H. Schmerl and W.T. Trotter, Critically indecomposable partially ordered sets, graphs, tournaments and other binary relational structures, *Discrete Math.* **113** (1993) 191–205.
[25] L.N. Shevrin and N.D. Filippov, Partially ordered sets and their comparability graphs, *Siberian Math. J.* **11** (1970) 497–509.

[26] J.B. Sidney, Optimal sequencing by modular decomposition, *Oper. Res.* **34** (1986) 606–626.
[27] J. Spinrad, P_4 trees and substitution decomposition, *Discrete Appl. Math.* **39** (1992) 263–291.
[28] J. Spinrad, On comparability and permutation graphs, *SIAM J. Comput.* **14** (1985) 658–670.
[29] J. Spinrad and J. Valdes, Recognition and isomorphism of two-dimensional partial orders, *Proc. 10th Coll. on Automata, Languages, and Programming,* Lecture Notes in Computer Science, Vol. **154** (Springer, Berlin, 1983) 676–686.
[30] K. Strassner, Zur Strukturtheorie endlicher nichtdeterministischer Automaten I, Zum Verband der 1-Kongruenzen von endlichen Relationalsystemen, *J. Inform. Process. Cybernetics (EIK)* **17** (1981) 113–120.
[31] K. Strassner, Zur Strukturtheorie endlicher nichtdeterministischer Automaten II, Gesteuerte Summen und Steuerbäume nichtdeterministischer Automaten, *J. Inform. Process. Cybernetics (EIK)* **17** (1981) 511–522.
[32] J. Valdes, R.E. Tarjan and E.L. Lawler, The recognition of series-parallel digraphs, *SIAM J. Comput.* **11** (1982) 299–313.