High-Performance Embedded Systems-on-a-Chip

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Lecture 5: Guibas-Kung-Thompson Array

Outline

- Projects Administrivia
- Manipulating Polyhedra & SRE's
- Optimal Parenthesization Array [GKT 79]

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Projects already “taken”

- Monica Chawathe & Charllie Ross: Interface generation
- Gautam Gupta: Scheduling/Serializing Reductions
- Dae-Kyoo Kim & Eunjee Song Optimal Prenthesization (protein folding)
- Howard Porter & Stacey Secatch Custom caches
- Lakshmi Renganarayana: Non-systolic SRE’s
- Jian-Pin Yang & Jhongjun Yang SOR Kernels

Manipulating Polyhedra and SRE's

Recap: Parameterized Polyhedra:

$$\{z \in \mathcal{Z}^n \mid Qz \geq q\}$$

Dual (alternative, equivalent) representation (vertices/rays):

$$\{z \in \mathcal{Z}^n \mid z = a^T V + b^T R;\ a_i, b_i \geq 0; \sum_i a_i = 1\}$$

Both are useful.

Standard (self-dual) algorithm to transform one to the other.

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Preimage by (any) affine function

$$\begin{aligned}
\mathcal{A}(z) &\equiv \mathcal{Z}^n \longrightarrow \mathcal{Z}^m \;:\; z \mapsto Az + a \\
\mathcal{P} &\equiv \text{polyhedron} \subseteq \mathcal{Z}^m \\
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&\equiv \{z \in \mathcal{Z}^n \,|\, \mathcal{A}(z) \in \mathcal{P}\}
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Result is a polyhedron with constraints $\langle QA, q - Qa \rangle$

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Image by (unimodular) affine function

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Practical hint: it is also ***preimage*** by $\mathcal{A}^{-1}$

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Recap: Change of Basis

To obtain an equivalent SRE on applying a *CoB transformation* by $\mathcal{T}$ to a variable X of an SRE:

$$X(z) \quad = \quad \left\{ \begin{array}{ccc} & \vdots & \\ D_i^X & : & g_i(\ldots Y(f(z))\ldots) \\ & \vdots & \end{array} \right.$$

Transformation Rules

- Replace each D_i^X by $\mathcal{T}(D_i^X)$, its image by $\mathcal{T}$.
- In the occurrences of a variable on the rhs of the equation for X, replace the dependency f by $f \circ \mathcal{T}^{-1}$, the composition[a] of f and $\mathcal{T}^{-1}$.
- In all occurrences of the variable X on the rhs of *any* equation, replace the dependency f by $\mathcal{T} \circ f$.

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[a]Note: function composition is rt associative: $(g \circ h)(z) = g(h(z))$.

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Optimal Parenthesization Array

Compute $C[1, n+1]$, where, for $1 \leq i < j \leq n+1$

$$C[i,j] = \begin{cases} i+1 = j & : \; f'(i,j) \\ i+1 < j & : \; \min_{i<k<j} (C[i,k] + C[k,j] + f(i,j,k)) \end{cases}$$

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Alternatively (à la Cormen et al) compute $C[1, n]$,

where, for $1 \leq i \leq j \leq n$,

$$C[i,j] = \begin{cases} i = j \;:\; 0 \\ i < j \;:\; \min\limits_{i \leq k < j} (C[i,k] + C[k+1,j] + f(i,j,k)) \end{cases}$$

Exercise

Prove that the two are equivalent (hint: substitute for the $i = j + 1$ case in the second one and “simplify”).

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Actual equation (in GKT): $f(i, j, k)$ is *independent of* k: compute $C[1, n+1]$, where, for $1 \leq i < j \leq n+1$

$$C[i,j] = w_{i,j} + \min_{i<k<j} (C[i,k] + C[k,j])$$

Observations/Questions

- Total volume of computation: $\approx \frac{1}{6}n^3$
- Total I/O volume : $\approx \frac{1}{2}n^2$ (actually n^2)
- Triangular array: PE $[i, j]$ computes $C[i, j]$

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- It is then sent horizontally and vertically, travelling at a rate of one PE per cycle for exactly $j - i$ more cycles.
- Afterwards, it travels at a slower rate of one PE every *two* cycles.