Chapter 5 Quick Review

1. Set : Collection of distinquishable objects
2. Bag : Collection with possible duplicates
3. Natural numbers are countably infinite
4. Reals are uncountable infinite

What about values between 0 and 1?
What is the power set?

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Equivalence Relations

Reflexive aRa
Symmetric aRb iff bRa
Transitive aRb & bRc $\Rightarrow$ aRc
An equivalence Relation forms a partion over a set A

Slide Lecture 5 –2

A Function is a binary relation on sets A $\times$ B such that $\forall a \in A$ there exists one $b \in B$ such that $(a, b) \in F$

b is the image of a under f

An Injection is a function that is one-to-one: distinct arguments to f produce distinct values.

A surjection is a function that is onto: every value is B is in the range of f and appears as the value of f for some argument.

$f(n) = \lfloor n/2 \rfloor$ is onto but not one-to-one

$f(n) = 2n$ is not onto for N but is one-to-one

A bijection is onto and one-to-one

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Chapter 6 Quick Review

$C(n, k) = \frac{n!}{k!(n-k)!}$ N Choose K

It is symetric $C(n, k) = C(n, n-k)$

Given the binomial coefficients $(x + u)^n = \sum_{k=0}^{n} C(n, k) x^k y^{n-k}$

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When $x = y = 1$ then $2^n = \sum_{k=0}^{n} C(n,k)$

0000	C(n,k)	1
0001		
0010	C(n,1)	4
0100		
1000		
0011		
0101		
1001	C(n,2)	6
0110		
1010		
1100		
1110		
1101	C(n,3)	4
1011		
0111		
1111	C(n,4)	1

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Bounds for Combinations

$$\begin{aligned} C(n,k) &= \frac{n(n-1)(n-2)...(n-k+1)}{k(k-1)...1} \\ &= (\frac{n}{k})(\frac{n-1}{k-1})...(\frac{n-k+1}{1}) \\ &\geq (\frac{n}{k})^k \end{aligned}$$

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$$C(n,k) = \frac{n(n-1)...(n-k+1)}{k(k-1)...1} \leq \frac{n^k}{k!}$$

Using Stirlings Approximation $K! \geq (K/e)^K$

Thererfore $C(n,k) \leq \frac{n^k}{K!} \leq \frac{n^K}{(K/e)^K} = \left(\frac{en}{k}\right)^K$

By induction one can also show

$C(n,k) \geq \frac{n^n}{k^k(n-k)^{n-k}}$

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Conditional Probability

The probability of event A given event B

$$Pr\{A|B\} = \frac{Pr\{A \cap B\}}{Pr\{B\}}$$

$A \cap B$ ... A and B both occur $Pr\{B\} \neq 0$

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Independence holds if

$Pr\{A \cap B\} = Pr\{A\}Pr\{B\}$ which implies $Pr\{A|B\} = PrA$ assuming $Pr\{B\} \neq 0$

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Bayes's Theorem

$Pr\{A|B\} = \frac{Pr\{A \cap B\}}{Pr\{B\}}$

implies

$$\begin{aligned} Pr\{A \cap B\} &= Pr\{B\}Pr\{A|B\} \\ &= Pr\{A\}Pr\{B|A\} \end{aligned}$$

Therefore

$Pr\{A|B\} = \frac{Pr\{A\}Pr\{B|A\}}{Pr\{B\}}$

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Also

$$\begin{aligned} Pr\{B\} &= Pr\{B \cap A\} + Pr\{B \cap \bar{A}\} \\ &= Pr\{A\}Pr\{B|A\} + Pr\{\bar{A}\}Pr\{B|\bar{A}\} \end{aligned}$$

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Example : Assume we are building chips that randomly generate 0 or 1. We know bad chips that output only 1 occur in 1 of 100 cases. A chip is tested and generates 1111. What is the probability this is a bad chip?

A ... Probability bad chip is picked
B ... Probability of output = 1111

$$\begin{aligned} Pr\{A|B\} &= \frac{Pr\{A\}Pr\{B|A\}}{Pr\{A\}Pr\{B|A\} + Pr\{\bar{A}\}Pr\{B|\bar{A}\}} \\ &= \frac{\frac{1}{100} \times 1}{\frac{1}{100} \times 1 + \frac{99}{100}\frac{1}{16}} \\ &= \frac{.1}{.161875} = .6177 \end{aligned}$$

How many times shoud we test to be 99% sure?

Slide Lecture 5 –12

Geometric and Binomial Distributions

A coin flip is a Bernoilli trial: an experiment with 2 outcomes with probabilites p and q where $(q = 1 - p)$.

$p =$ probability of success
$q = 1 - p =$ probabilitiy of failure

Let x be the number of trials neeed to obtain a success,

$$x \in \{1, 2, 3...\}, \quad k \geq 1, \quad Pr\{x = k\} = q^{k-1}p$$

because there are $(k-1)$ failures and success on the k^{th} test.

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Expected value of $E(x)$ for $p < 1$

$$\begin{aligned} E[x] &= \sum_{k=1}^{\infty} kq^{k-1}p \\ &= \frac{p}{q}\sum_{k=0}^{\infty} kq^k \\ &= \frac{p}{q}\frac{q}{(1-q)^2} \\ &= \frac{p}{p^2} = \frac{1}{p} \end{aligned}$$

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Another point of view

A slot machine gives a prize with probability p. You must pay \$1 to play. If you play an infinite number of games then

$E[x] = \sum_{k=1}^{\infty} kq^{k-1}p = \frac{1}{p}$ is the average cost of winning a prize.
Variance $= \frac{q}{p^2}$

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Binomial Distribution

After N Bernoulli trials, Let x be the number of successes.

What is the probability $X = k$?

$Pr\{x = k\} = C(n, k)p^k q^{n-k}$

C(n,k)... ways to pick k of n objects/trials (e.g. number of binary strings of length N with K bits set to 1).

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A Binomial Distribution

$b(k;n,p) = C(n,k)p^k(1-p)^{n-k}$ since $p+q=1$

$\sum_{k=0}^{n} b(k;n,p) = 1$

e.g. Probability of generating 1 of all possible binary strings must = 1

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Tails of the Binomial Distribution

$$\begin{aligned} Pr\{X \geq k\} &= \sum_{i=k}^{n} b(i;n,p) \\ &= \sum_{i=0}^{n-k} b(k+i;n,p) \\ &= \sum_{i=0}^{n-k} C(n,k+1)p^{k+i}(1-p)^{n-(k+i)} \end{aligned}$$

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Note $C(n, k+i) \geq C(n,k)C(n-k,i)$ Thus:

$$
\begin{aligned}
Pr\{X \geq k\} &\leq \sum_{i=0}^{n-k} C(n,k)C(n-k,i)p^{k+i}(1-p)^{n-(k+i)} \\
&= C(n,k)\sum_{i=0}^{n-k} C(n-k,i)p^{k+i}(1-p)^{n-(k+i)} \\
&= C(n,k)p^k \sum_{i=0}^{n-k} C(n-k,i)p^{i}(1-p)^{n-k-i} \\
&= C(n,k)p^k \sum_{i=0}^{n-k} b(i;n-k,p) \\
&= C(n,k)p^k
\end{aligned}
$$

$Pr\{X \geq k\} \leq C(n,k)p^k$

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Probability of at most K successes (or at least n-k failures)

$$
\begin{aligned}
Pr\{X \leq k\} &= \sum_{i=0}^{k} b(i;n,p) \\
&\leq C(n,n-k)(1-p)^{n-k} \\
&= C(n,k)(1-p)^{n-k}
\end{aligned}
$$

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The Birthday Paradox

How many people must be in a room before there is a good chance 2 were born on the same day?

Index the people 1, 2,...,k
Assume all years are 365 days(n)
Probability person 1's birthday is unique (n/n)
Probability that person 2's birthday is unique (n-1/n)
Person k : (n-k/n)

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$Pr\{B_k\} = n/n * n - 1/n * n - 2/n ... n - (k-1)/n =$
$1(1 - 1/n)(1 - 2/n)...(1 - k - 1/n)$
Axiom : $(1 + x \leq e^x) and (1 - x) \leq e^{-x}$
Therefore:
$Pr\{B_k\} \leq (e^{-1/n})(e^{-2/n})...(e^{-(k-1)/n}) = e^{-\sum_{i=1}^{k-1} i/n}$

So $Pr\{B_k\} \leq e^{-\sum_{i=1}^{k-1} 1/n}$ Consider $e^{-\sum_{i=1}^{k-1} 1/n} = 1/2$ and $Pr\{B_k\} \leq 1/2$

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Now

$$
\begin{aligned}
e^{-\sum_{i=1}^{k-1} 1/n} &= -1/n \sum_{i=1}^{k-1} i \\
&= -1/nk(k-1)/2 \\
&= -k(k-1)/2n
\end{aligned}
$$

therefore $e^{-k(k-1)/2n}$

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Consider

$$
\begin{aligned}
e^{-k(k-1)/2n} &= 1/2 \\
-k(k-1)/2n &= ln(1/2) \\
Pr\{B_k\} & leq \ 1/2
\end{aligned}
$$

when

$$
\begin{aligned}
-k(k-1)/2n &\leq \ln(1/2) \\
\ln(1/2) &= -\ln(2)
\end{aligned}
$$

solve for $k(k-1) \geq 2n \ln(2)$

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$\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

therefore $Pr\{B_k\}$ when

$k \geq (1 + \sqrt{1 + (8ln2)n})/2$

For $n = 365, k \geq 23$

Slide Lecture 5 –25

Assume local optima are uniformly distributed in a search space. Statistically I find on average I get 2 duplicate optima after 31 samples. How many local optima are present.

Approximates 669

(note bounds are in the other direction)

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