Sets and Functions
(Rosen, Sections 2.1, 2.2, 2.3)

TOPICS

• Discrete math
• Set Definition
• Set Operations
• Tuples

Discrete Math at CSU (Rosen book)

- CS 160 or CS122
  - Sets and Functions
  - Propositions and Predicates
  - Inference Rules
  - Proof Techniques
  - Program Verification
- CS 161
  - Counting
  - Induction proofs
  - Recursion
- CS 200
  - Algorithms
  - Relations
  - Graphs
Why Study Discrete Math?

- Digital computers are based on discrete units of data (bits).
- Therefore, both a computer’s
  - structure (circuits) and
  - operations (execution of algorithms)
  can be described by discrete math
- A generally useful tool for rational thought! Prove your arguments.

What is ‘discrete’?

- Consisting of distinct or unconnected elements, not continuous (calculus)
- Helps us in Computer Science:
  - What is the probability of winning the lottery?
  - How many valid Internet address are there?
  - How can we identify spam e-mail messages?
  - How many ways are there to choose a valid password on our computer system?
  - How many steps are need to sort a list using a given method?
  - How can we prove our algorithm is more efficient than another?
Uses for Discrete Math in Computer Science

- Advanced algorithms & data structures
- Programming language compilers & interpreters.
- Computer networks
- Operating systems
- Computer architecture
- Database management systems
- Cryptography
- Error correction codes
- Graphics & animation algorithms, game engines, etc.…
- i.e., the whole field!

What is a set?

- An unordered collection of unique objects
  - \{1, 2, 3\} = \{3, 2, 1\} since sets are unordered.
  - \{a, b, c\} = \{b, c, a\} = \{c, b, a\} = \{a, b, c\} = \{a, c, b\}
  - \{2\}
  - \{on, off\}
  - \{\}
  - \{1, 2, 3\} = \{1, 1, 2, 3\} since elements in a set are unique
What is a set?

- Objects are called elements or members of the set.
- **Notation** \( \in \) means “a is an element of set B.”
- Lower case letters for elements in the set.
- Upper case letters for sets.

**If** \( A = \{1, 2, 3, 4, 5\} \) and \( x \in A \), what are the possible values of \( x \)?

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What is a set?

- **Infinite Sets** (without end, unending)
  - \( N = \{0, 1, 2, 3, \ldots\} \) is the Set of natural numbers.
  - \( Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \) is the Set of integers.
  - \( Z^+ = \{1, 2, 3, \ldots\} \) is the Set of positive integers.

- **Finite Sets** (limited number of elements)
  - \( V = \{a, e, i, o, u\} \) is the Set of vowels.
  - \( O = \{1, 3, 5, 7, 9\} \) is the Set of odd #’s < 10.
  - \( F = \{a, 2, Fred, New Jersey\} \)
  - Boolean data type used frequently in programming
    - \( B = \{0, 1\} \)
    - \( B = \{false, true\} \)
  - **Seasons** = \{spring, summer, fall, winter\}
  - **ClassLevel** = \{Freshman, Sophomore, Junior, Senior\}
What is a set?

- Infinite vs. finite
  - If finite, then the number of elements is called the *cardinality*, denoted $|S|$.
    - $V = \{a, e, i, o, u\}$  $|V| = 5$
    - $F = \{1, 2, 3\}$  $|F| = 3$
    - $B = \{0, 1\}$  $|B| = 2$
    - $S = \{\text{spring, summer, fall, winter}\}$  $|S| = 4$
    - $A = \{a, a, a\}$  $|A| = 1$

Example sets

- Alphabet
- All characters
- Booleans: true, false
- Numbers:
  - $N = \{0, 1, 2, 3\ldots\}$  Natural numbers
  - $Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$  Integers
  - $Q = \{p / q \mid p \in Z, q \in Z, q \neq 0\}$  Rationals
  - $R$, Real Numbers
- Note that:
  - $Q$ and $R$ are not the same. $Q$ is a *subset* of $R$.
  - $N$ is a subset of $Z$. 
Example: Set of Bit Strings

- A bit string is a sequence of zero or more bits.
- A bit string's length is the number of bits in the string.
- A set of all bit strings $s$ of length 3 is
  - $S = \{000, 001, 010, 011, 100, 101, 110, 111\}$

What is a set?

- **Defining a set:**
  - **Option 1:** List the members
  - **Option 2:** Use a set builder that defines set of $x$ that hold a certain characteristic
  - **Notation:** \( \{ x \in S \mid \text{characteristic of } x \} \)
  - **Examples:**
    - $A = \{ x \in \mathbb{Z}^+ \mid x \text{ is prime} \}$ – set of all prime positive integers
    - $O = \{ x \in \mathbb{N} \mid x \text{ is odd and } x < 10000 \}$ – set of odd natural numbers less than 10000
Equality

- Two sets are *equal* if and only if (iff) they have the same elements.
- We write $A = B$ when for all elements $x$, $x$ is a member of the set $A$ iff $x$ is also a member of $B$.
  - Notation: $\forall x \{ x \in A \leftrightarrow x \in B \}$
  - For all values of $x$, $x$ is an element of $A$ if and only if $x$ is an element of $B$

Set Operations

- Operations that take as input sets and have as output sets
- Operation1: *Union*
  - The union of the sets $A$ and $B$ is the set that contains those elements that are either in $A$ or in $B$, or in both.
  - Notation: $A \cup B$
  - Example: union of $\{1,2,3\}$ and $\{1,3,5\}$ is?
Operation 2: Intersection

- The intersection of sets $A$ and $B$ is the set containing those elements in both $A$ and $B$.
- Notation: $A \cap B$
- Example: $\{1,2,3\}$ intersection $\{1,3,5\}$ is?
- The sets are disjoint if their intersection produces the empty set.

Operation 3: Difference

- The difference of $A$ and $B$ is the set containing those elements that are in $A$ but not in $B$.
- Notation: $A \setminus B$
- Aka the complement of $B$ with respect to $A$
- Example: $\{1,2,3\}$ difference $\{1,3,5\}$ is?
- Can you define Difference using union, complement and intersection?
Operation 3: Complement

- The complement of set $A$ is the complement of $A$ with respect to $U$, the universal set.
- Notation: $\overline{A}$
- Example: If $\mathbb{N}$ is the universal set, what is the complement of $\{1, 3, 5\}$?
  
  Answer: $\{0, 2, 4, 6, 7, 8, \ldots\}$

Venn Diagram

- Graphical representation of set relations:

```
  A  B

  U
```
Idsentities

<table>
<thead>
<tr>
<th>Identity</th>
<th>( A \cup \emptyset = A, A \cap U = A )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Commutative</td>
<td>( A \cup B = B \cup A, A \cap B = B \cap A )</td>
</tr>
<tr>
<td>Associative</td>
<td>( A \cup (B \cup C) = (A \cup B) \cup C, A \cap (B \cap C) = (A \cap B) \cap C )</td>
</tr>
<tr>
<td>Distributive</td>
<td>( A \cap (B \cup C) = (A \cap B) \cup (A \cap C), A \cup (B \cap C) = (A \cup B) \cap (A \cup C) )</td>
</tr>
<tr>
<td>Complement</td>
<td>( A \cup \overline{A} = U, A \cap \overline{A} = \emptyset )</td>
</tr>
</tbody>
</table>

Subset

- The set \( A \) is said to be a subset of \( B \) iff for all elements \( x \) of \( A \), \( x \) is also an element of \( B \).
  
  \textit{But not necessarily the reverse…}

- Notation: \( A \subseteq B \quad \forall x \{ x \in A \rightarrow x \in B \} \)
  
  - Unidirectional implication

- \( \{1,2,3\} \subseteq \{1,2,3\} \)
- \( \{1,2,3\} \subseteq \{1,2,3,4,5\} \)
- What is the cardinality between sets if \( A \subseteq B \)?

  \textbf{Answer:} \( |A| \leq |B| \)
Subset

- **Subset** is when a set is contained in another set. Notation: \( \subseteq \)
- **Proper subset** is when \( A \) is a subset of \( B \), but \( B \) is not a subset of \( A \). Notation: \( \subset \)
  - \( \forall x ((x \in A) \rightarrow (x \in B)) \land \exists x ((x \in B) \land (x \notin A)) \)
  - All values \( x \) in set \( A \) also exist in set \( B \)
  - … but there is at least 1 value \( x \) in \( B \) that is not in \( A \)
  - \( A = \{1,2,3\}, B = \{1,2,3,4,5\} \)
  - \( A \subset B \), means that \( |A| < |B| \).

Empty Set

- **Empty set** has no elements and therefore is the subset of all sets. \{\} Alternate Notation: \( \emptyset \)
- Is \( \emptyset \subseteq \{1,2,3\} \)? - Yes!
- The cardinality of \( \emptyset \) is zero: \( |\emptyset| = 0 \).
- Consider the set containing the empty set: \( \{\emptyset\} \).
- Yes, this is indeed a set: \( \emptyset \in \{\emptyset\} \) and \( \emptyset \subseteq \{\emptyset\} \).
Set Theory - Definitions and notation

• Quiz time:
  • A = \{ x \in \mathbb{N} \mid x \leq 2000 \}  What is |A| = 2001?
  • B = \{ x \in \mathbb{N} \mid x \geq 2000 \}  What is |B| = Infinite!
  • Is \{x\} \subseteq \{x\}? Yes
  • Is \{x\} \in \{x,\{x\}\}? Yes
  • Is \{x\} \subseteq \{x,\{x\}\}? Yes
  • Is \{x\} \in \{x\}? No

Powerset

The powerset of a set is the set containing all the subsets of that set.

Notation:  \(P(A)\) is the powerset of set A.

Fact:  \(|P(A)| = 2^{|A|}\).
  • If A = \{x, y\}, then  \(P(A) = \{\emptyset, \{x\}, \{y\}, \{x,y\}\}\)
  • If S = \{a, b, c\}, what is \(P(S)\)?
Powerset example

- Number of elements in powerset = \(2^n\) where \(n = \#\) elements in set
  - \(S\) is the set \(\{a, b, c\}\), what are all the subsets of \(S\)?
    - \(\{\}\) – the empty set
    - \(\{a\}, \{b\}, \{c\}\) – one element sets
    - \(\{a, b\}, \{a, c\}, \{b, c\}\) – two element sets
    - \(\{a, b, c\}\) – the original set
  
  and hence the power set of \(S\) has \(2^3 = 8\) elements:

\[
\{\{\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}\}
\]

Why sets?

- Programming - Recall a class… it is the set of all its possible objects.
- We can restrict the type of an object, which is the set of values it can hold.
  - Example: Data Types
    - \texttt{int} set of integers (finite)
    - \texttt{char} set of characters (finite)
  
- Is \(\mathbb{N}\) the same as the set of integers in a computer?
Order Matters

- What if order matters?
  - Sets disregard ordering of elements
  - If order is important, we use *tuples*
  - If order matters, then are duplicates important too?

Tuples

- Order matters
- Duplicates matter
- Represented with parens ( )
- Examples
  - \((1, 2, 3) \neq (3, 2, 1) \neq (1, 1, 2, 3, 3)\)
  - \(\left(a_1, a_2, ..., a_n\right)\)
Tuples

- The ordered \( n \)-tuple \((a_1,a_2,\ldots,a_n)\) is the ordered collection that has \(a_1\) as its first element, \(a_2\) as its second element, … and \(a_n\) as its \(n\)th element.

- An ordered pair is a 2-tuple.

- Two ordered pairs \((a,b)\) and \((c,d)\) are equal iff \(a=c\) and \(b=d\) \(\text{(e.g. NOT if } a=d \text{ and } b=c\text{)}\).

- A 3-tuple is a triple; a 5-tuple is a quintuple.

In programming?

- Let’s say you’re working with three integer values, first is the office room # of the employee, another is the # years they’ve worked for the company, and the last is their ID number.

- Given the following set \(\{320, 13, 4392\}\), how many years has the employee worked for the company?

- What if the set was \(\{320, 13, 4392\}\)? Doesn’t \(\{320, 13, 4392\} = \{320, 4392, 13\}\) ?

- Given the 3-tuple \((320, 13, 4392)\) can we identify the number of years the employee worked?
Why?

- Because ordered n-tuples are found as lists of arguments to functions/methods in computer programming.
- Create a mouse in a position (2, 3) in a maze: `new Mouse(2, 3)`
- Can we reverse the order of the parameters?
- From Java, `Math.min(1, 2)`

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Cartesian Product of Two Sets

- Let A and B be sets. The Cartesian Product of A and B is the set of all ordered pairs (a,b), where \( b \in B \) and \( a \in A \)
- Cartesian Product is denoted \( A \times B \).
- Example: \( A=\{1,2\} \) and \( B=\{a,b,c\} \). What is \( A \times B \) and \( B \times A \)?
Cartesian Product

- $A = \{a, b\}$
- $B = \{1, 2, 3\}$
- $A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$
- $B \times A = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$

Functions in CS

- Function = mappings or transformations
- Examples
  - $f(x) = x$
  - $f(x) = x + 1$
  - $f(x) = 2x$
  - $f(x) = x^2$
Function Definitions

• A function $f$ from sets $A$ to $B$ assigns exactly one element of $B$ to each element of $A$.

• Example: the floor function

<table>
<thead>
<tr>
<th>Domain</th>
<th>Codomain</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.4</td>
<td>1</td>
</tr>
<tr>
<td>1.6</td>
<td>2</td>
</tr>
<tr>
<td>5.0</td>
<td>3</td>
</tr>
<tr>
<td>4.8</td>
<td>4</td>
</tr>
<tr>
<td>2.3</td>
<td>5</td>
</tr>
</tbody>
</table>

Range: $\{1, 2, 4, 5\}$

What's the difference between codomain and range?

Range contains the codomain values that $A$ maps to.

Function Definitions

• In Programming
  – Function header definition example

```c
int floor( float real)
{
}
```

• Domain = $\mathbb{R}$
• Codomain = $\mathbb{Z}$
Other Functions

• The **identity** function, $f_{ID}$, on $A$ is the function where: $f_{ID}(x) = x$ for all $x$ in $A$.
  
  $A = \{a,b,c\}$ and $f(a) = a$, $f(b) = b$, $f(c) = c$

- **Successor function**, $f_{succ}(x) = x+1$, on $\mathbb{Z}$
  
  - $f(1) = 2$
  - $f(-17) = -16$
  - $f(a)$ Does NOT map to $b$

- **Predecessor function**, $f_{pred}(x) = x-1$, on $\mathbb{Z}$
  
  - $f(1) = 0$
  - $f(-17) = -18$

• $f_{NEG}(x) = -x$, also on $\mathbb{R}$ (or $\mathbb{Z}$), maps a value into the negative of itself.

• $f_{SQ}(x) = x^2$, maps a value, $x$, into its square, $x^2$.

• The **ceiling** function: $ceil(2.4) = 3$. 
Functions in CS

• What are ceiling and floor useful for?
  – Data stored on disk are represented as a string of bytes. Each byte = 8 bits. How many bytes are required to encode 100 bits of data?

Need smallest integer that is at least as large as 100/8

100/8 = 12.5
But we don’t work with ½ a byte.
So we need 13 bytes

What is NOT a function?

• Consider $f_{\text{SQRT}}(x)$ from $\mathbb{Z}$ to $\mathbb{R}$.
• This does not meet the given definition of a function, because $f_{\text{SQRT}}(16) = \pm 4$.
• In other words, $f_{\text{SQRT}}(x)$ assigns exactly one element of $\mathbb{Z}$ to two elements of $\mathbb{R}$.

Note that the convention is that $\sqrt{x}$ is always the positive value.
$f_{\text{SQRT}}(x) = \pm \sqrt{x}$

No Way!
Say it ain’t so!!
1 to 1 Functions

- A function $f$ is said to be one-to-one or injective if and only if $f(a) = f(b)$ implies that $a = b$ for all $a$ and $b$ in the domain of $f$.
- Example: the square function from $\mathbb{Z}^+$ to $\mathbb{Z}^+$

![Diagram of square function]

1 to 1 Functions, cont.

- Is square from $\mathbb{Z}$ to $\mathbb{Z}$ an example?
  - NO!
  - Because $f_{\text{SQ}}(-2) = 4 = f_{\text{SQ}}(+2)$!
- Is floor an example?
  INCONCEIVABLE!!
- Is identity an example?
  Unique at last!!
Increasing Functions

- A function $f$ whose domain and co-domain are subsets of the set of real numbers is called *increasing* if $f(x) \leq f(y)$ and *strictly increasing* if $f(x) < f(y)$, whenever
  - $x < y$ and
  - $x$ and $y$ are in the domain of $f$.
- Is *floor* an example?
  
  1. $1.5 < 1.7$ and $\text{floor}(1.5) = 1 = \text{floor}(1.7)$
  2. $1.2 < 2.2$ and $\text{floor}(1.2) = 1 < 2 = \text{floor}(2.2)$
- Is *square* an example?
  
  When mapping $Z$ to $Z$ or $R$ to $R$:
  - $\text{square}(-2) = 4 > 1 = \text{square}(1)$ yet $-2 < 1$

How is Increasing Useful?

- Most programs run longer with larger or more complex inputs.
- Consider looking up a telephone number in the paper directory...
Cartesian Products and Functions

• A function with multiple arguments maps a Cartesian product of inputs to a codomain.

• Example:
  – `Math.min` maps $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$
    ```java
    int minVal = Math.min(23, 99);
    ```
  – `Math.abs` maps $\mathbb{Q}$ to $\mathbb{Q}^+$
    ```java
    int absVal = Math.abs(-23);
    ```

Quiz Check

• Is the following an increasing function?
  - $\mathbb{Z} \rightarrow \mathbb{Z}$ $f(x) = x + 5$
  - $\mathbb{Z} \rightarrow \mathbb{Z}$ $f(x) = 3x - 1$