Sets
(Rosen, Sections 2.1,2.2)

TOPICS
• Discrete math
• Set Definition
• Set Operations
• Tuples

Why Study Discrete Math?

• Digital computers are based on discrete units of data (bits).
• Therefore, both a computer’s structure (circuits) and operations (execution of algorithms) can be described by discrete math
• A generally useful tool for rational thought! Prove your arguments.

What is ‘discrete’?

• Consisting of distinct or unconnected elements, not continuous (calculus)
• Helps us in Computer Science:
  • What is the probability of winning the lottery?
  • How many valid Internet address are there?
  • How can we identify spam e-mail messages?
  • How many ways are there to choose a valid password on our computer system?
  • How many steps are needed to sort a list using a given method?
  • How can we prove our algorithm is more efficient than another?

Uses for Discrete Math in Computer Science

• Advanced algorithms & data structures
• Programming language compilers & interpreters.
• Computer networks
• Operating systems
• Computer architecture
• Database management systems
• Cryptography
• Error correction codes
• Graphics & animation algorithms, game engines, etc.…
• i.e., the whole field!
What is a set?

- **An unordered collection of objects**
  - \( \{1, 2, 3\} = \{3, 2, 1\} \) since sets are unordered.
  - \( \{a, b, c\} = \{b, c, a\} = \{c, b, a\} = \{a, c, b\} \)
  - \( \{2\} \)
  - \{on, off\}
  - \( \{} \)

Objects are called **elements** or **members** of the set.

**Notation** \( \in \)
- \( a \in B \) means “\( a \) is an element of set \( B \).”
- Lower case letters for elements in the set
- Upper case letters for sets
- If \( A = \{1, 2, 3, 4, 5\} \) and \( x \in A \), what are the possible values of \( x \)?

**Infinite Sets (without end, unending)**
- \( N = \{0, 1, 2, 3, \ldots\} \) is the Set of natural numbers
- \( Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \) is the Set of integers
- \( Z^+ = \{1, 2, 3, \ldots\} \) is the Set of positive integers

**Finite Sets (limited number of elements)**
- \( V = \{a, e, i, o, u\} \) is the Set of vowels
- \( O = \{1, 3, 5, 7, 9\} \) is the Set of odd #’s < 10
- \( F = \{a, 2, Fred, New Jersey\} \)
- Boolean data type used frequently in programming
  - \( B = \{0, 1\} \)
  - \( B = \{false, true\} \)
- Seasons = \{spring, summer, fall, winter\}
- ClassLevel = \{Freshman, Sophomore, Junior, Senior\}

**Infinite vs. finite**
- If finite, then the number of elements is called the **cardinality**, denoted \( |S| \)
  - \( V = \{a, e, i, o, u\} \) \( |V| = 5 \)
  - \( F = \{1, 2, 3\} \) \( |F| = 3 \)
  - \( B = \{0, 1\} \) \( |B| = 2 \)
  - \( S = \{spring, summer, fall, winter\} \) \( |S| = 4 \)
Example sets

- Alphabet
- All characters
- Booleans: true, false
- Numbers:
  - \(N = \{0,1,2,3\ldots\}\) - Natural numbers
  - \(Z = \{\ldots,-2,-1,0,1,2\ldots\}\) - Integers
  - \(Q = \{p/q \mid p \in Z, q \in Z, q \neq 0\}\) - Rationals
  - \(R\), Real Numbers
- Note that:
  - \(Q\) and \(R\) are not the same. \(Q\) is a subset of \(R\).
  - \(N\) is a subset of \(Z\).

What is a set?

- Defining a set:
  - Option 1: List the members
  - Option 2: Use a \textit{set builder} that defines set of \(x\) that hold a certain characteristic
  - Notation: \(\{x \in S \mid \text{characteristic of } x\}\)
  - Examples:
    - \(A = \{x \in Z^+ \mid x \text{ is prime}\}\) – set of all prime positive integers
    - \(O = \{x \in N \mid x \text{ is odd and } x < 10000\}\) – set of odd natural numbers less than 10000

Equality

- \(A = B\) is used to show set equality
- Two sets are \textit{equal} when they have exactly the same elements
- Thus for all elements \(x\), \(x\) belongs to \(A\) \textit{if and only if} \((\text{iff}) \ x\) also belongs to \(B\)
- The if and only is a bidirectional implication that we will study later
Set Operations: Union

- Operations that take as input sets and have as output sets
- The union of the sets A and B is the set that contains those elements that are either in A or in B, or in both.
  - Notation: $A \cup B$
  - Example: union of \{1, 2, 3\} and \{1, 3, 5\} is?

Answer: \{1, 2, 3, 5\}

Set Operations: Intersection

- The intersection of sets A and B is the set containing those elements in both A and B.
- Notation: $A \cap B$
- The sets are disjoint if their intersection produces the empty set.
- Example: \{1, 2, 3\} intersection \{1, 3, 5\} is?

Answer: \{1, 3\}

Set Operations: Difference

- The difference of A and B is the set of elements that are in A but not in B.
- Notation: $A - B$
- Aka the complement of B with respect to A
- Can you define difference using union, complement and intersection?
- Example: \{1, 2, 3\} difference \{1, 3, 5\} is?

Answer: \{2\}

Set Operations: Complement

- The complement of set A is the complement of A with respect to U, the universal set.
- Notation: $\overline{A}$
- Example: If N is the universal set, what is the complement of \{1, 3, 5\}?
  Answer: \{0, 2, 4, 6, 7, 8, ...\}
Identities

- **Identity**
  \[ A \cup \emptyset = A, A \cap U = A \]

- **Commutative**
  \[ A \cup B = B \cup A, \; A \cap B = B \cap A \]

- **Associative**
  \[ A \cup (B \cup C) = (A \cup B) \cup C, A \cap (B \cap C) = (A \cap B) \cap C \]

- **Distributive**
  \[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C), A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \]

- **Complement**
  \[ A \cup \overline{A} = U, A \cap \overline{A} = \emptyset \]

Subsets

- The set \( A \) is a subset of \( B \) iff for all elements \( x \) of \( A \), \( x \) is also an element of \( B \). *But not necessarily the reverse…*

- **Notation:** \( A \subseteq B \)
  - \( \{1,2,3\} \subseteq \{1,2,3\} \)
  - \( \{1,2,3\} \subseteq \{1,2,3,4,5\} \)
  - What is the relationship of the cardinality between sets if \( A \subseteq B \)? \( |A| \leq |B| \)

Empty Set

- **Empty set** has no elements and therefore is the subset of all sets: \( \{ \} \) or \( \emptyset \)

- Is \( \emptyset \subseteq \{1,2,3\} \)? - Yes!

- The cardinality of \( \emptyset \) is zero: \( |\emptyset| = 0 \).

- Consider the set containing the empty set: \( \{\emptyset\} \)

- Yes, this is indeed a set:
  \( \emptyset \in \{\emptyset\} \) and \( \emptyset \subseteq \{\emptyset\} \).
Set Theory

Quiz time:

• $A = \{ x \in \mathbb{N} \mid x \leq 2000 \}$  What is $|A|$?  2001
• $B = \{ x \in \mathbb{N} \mid x \geq 2000 \}$  What is $|B|$?  Infinite
• Is $\{ x \} \subseteq \{ x \}$?  Yes
• Is $\{ x \} \in \{ x \}$?  Yes
• Is $\{ x \} \subseteq \{ x, (x) \}$?  Yes
• Is $\{ x \} \in \{ x \}$?  No

Powerset

• The powerset of a set is the set containing all the subsets of that set.
• Notation: $P(A)$ is the powerset of set $A$.
• Fact: $|P(A)| = 2^{|A|}$.
  - If $A = \{ x, y \}$, then $P(A) = \{ \emptyset, \{ x \}, \{ y \}, \{ x, y \} \}$
  - If $S = \{ a, b, c \}$, what is $P(S)$?

Powerset example

• Number of elements in powerset $= 2^n$ where $n =$ # elements in set
  - $\{ \}$ – the empty set
  - $\{ a \}, \{ b \}, \{ c \}$ – one element sets
  - $\{ a, b \}, \{ a, c \}, \{ b, c \}$ – two element sets
  - $\{ a, b, c \}$ – the original set

and hence the power set of $S$ has $2^3 = 8$ elements:

$\emptyset, \{ a \}, \{ b \}, \{ c \}, \{ a, b \}, \{ a, c \}, \{ b, c \}, \{ a, b, c \}$

Example

• Consider binary numbers
  - E.g. 0101
• Let every bit position $\{1, \ldots,n\}$ be an item
  - Position $i$ is in the set if bit $i$ is 1
  - Position $i$ is not in the set if bit $i$ is 0
• What is the set of all possible N-bit numbers?
  - The powerset of $\{1, \ldots,n\}$. 
Why sets?

- Programming - Recall a class… it is the set of all its possible objects.
- We can restrict the type of an object, which is the set of values it can hold.
  - Example: Data Types
    - int: set of integers (finite)
    - char: set of characters (finite)
- Is \( \mathbb{N} \) the same as the set of integers in a computer?
- Is \( \mathbb{Q} \) or \( \mathbb{R} \) the same as the set of doubles in a computer?

Order Matters

- What if order matters?
  - Sets disregard ordering of elements
  - If order is important, we use tuples
  - If order matters, then are duplicates important too?

Tuples

- Order matters
- Duplicates matter
- Represented with parens ( )
- Examples
  - \((1, 2, 3) \neq (3, 2, 1) \neq (1, 1, 2, 3, 3)\)
  - \(\{a_1, a_2, \ldots, a_n\}\)

- The ordered \(n\)-tuple \((a_1, a_2, \ldots, a_n)\) is the ordered collection that has \(a_1\) as its first element, \(a_2\) as its second element \(\ldots\) and \(a_n\) as its \(n\)th element.
- An ordered pair is a 2-tuple.
- Two ordered pairs \((a, b)\) and \((c, d)\) are equal iff \(a = c\) and \(b = d\) (e.g. NOT if \(a = d\) and \(b = c\)).
- A 3-tuple is a triple; a 5-tuple is a quintuple.
Tuples

- In programming?
  - Let’s say you’re working with three integer values, first is the office room # of the employee, another is the # years they’ve worked for the company, and the last is their ID number.
  - Given the following set \{320, 13, 4392\}, how many years has the employee worked for the company?
  - What if the set was \{320, 13, 4392\}? Doesn’t \{320, 13, 4392\} = \{320, 4392, 13\}?
  - Given the 3-tuple \((320, 13, 4392)\) can we identify the number of years the employee worked?

Why?

- Because ordered n-tuples are found as lists of arguments to functions/methods in computer programming.
- Create a mouse in a position (2, 3) in a maze: `new Mouse(2, 3)`
- Can we reverse the order of the parameters?
- From Java, `Math.min(1, 2)`

Cartesian Product

- Let A and B be sets. The Cartesian Product of A and B is the set of all ordered pairs (a,b), where \(b \in B\) and \(a \in A\).
- Cartesian Product is denoted \(A \times B\).
- Example: \(A = \{1, 2\}\) and \(B = \{a, b, c\}\). What is \(A \times B\) and \(B \times A\)?

- A = \{a, b\}
- B = \{1, 2, 3\}
- \(A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}\)
- \(B \times A = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}\)