Functions
(Rosen, Section 2.3)

TOPICS

• Definition of Function
• Domain, Codomain, Range
• One-to-One Functions
• Increasing Functions

Functions in CS

• Function = mappings or transformations

• Examples
  
  \[ f(x) = x \]
  \[ f(x) = x + 1 \]
  \[ f(x) = 2x \]
  \[ f(x) = x^2 \]

Function Definitions

• A function \( f \) from sets \( A \) to \( B \) assigns exactly one element of \( B \) to each element of \( A \).

• Example: the \( \text{floor} \) function

  \[
  \begin{array}{c|c}
    \text{Domain} & \text{Codomain} \\
    \hline
    2.4 & 1 \\
    1.6 & 2 \\
    5.0 & 3 \\
    4.8 & 4 \\
    2.3 & 5 \\
  \end{array}
  \]

  Range: \( \{1, 2, 4, 5\} \)

What’s the difference between codomain and range?

Range contains the codomain values that \( A \) maps to

Function Definitions

• In Programming
  
  — Function header definition example

  ```
  int floor( float real )
  {
  }
  ```

  • Domain = \( \mathbb{R} \)
  • Codomain = \( \mathbb{Z} \)
Other Functions

- The identity function, $f_{ID}$, on $A$ is the function where: $f_{ID}(x) = x$ for all $x$ in $A$.
  - $A = \{a, b, c\}$ and $f(a) = a$, $f(b) = b$, $f(c) = c$

- Successor function, $f_{succ}(x) = x+1$, on $\mathbb{Z}$
  - $f(1) = 2$
  - $f(-17) = -16$
  - $f(a)$ Does NOT map to $b$

- Predecessor function, $f_{pred}(x) = x-1$, on $\mathbb{Z}$
  - $f(1) = 0$
  - $f(-17) = -18$

Other Functions

- $f_{NEG}(x) = -x$, also on $\mathbb{R}$ (or $\mathbb{Z}$), maps a value into the negative of itself.

- $f_{SQ}(x) = x^2$, maps a value, $x$, into its square, $x^2$.

- The ceiling function: $\text{ceil}(2.4) = 3$.

Functions in CS

- What are ceiling and floor useful for?
  - Data stored on disk are represented as a string of bytes. Each byte = 8 bits. How many bytes are required to encode 100 bits of data?

  **Need smallest integer that is at least as large as 100/8**
  
  $100/8 = 12.5$ But we don’t work with $\frac{1}{2}$ a byte.
  So we need 13 bytes

- What is NOT a function?
  - Consider $f_{SORT}(x)$ from $\mathbb{Z}$ to $\mathbb{R}$.
    - This does **not** meet the given definition of a function, because $f_{SORT}(16) = \pm 4$.
    - In other words, $f_{SORT}(x)$ assigns exactly **one** element of $\mathbb{Z}$ to **two** elements of $\mathbb{R}$.

  **No Way! Say it ain’t so!!**

  Note that the convention is that $\forall x$ is always the **positive value**. $f_{SORT}(x) = \sqrt{x}$
1 to 1 Functions

- A function \( f \) is said to be one-to-one or injective if and only if \( f(a) = f(b) \) implies that \( a = b \) for all \( a \) and \( b \) in the domain of \( f \).
- Example: the square function from \( \mathbb{Z} \) to \( \mathbb{Z} \):

```
  1  2  3  4 ...
  2  3  4 ...
  3  ...
  4 ...
  ...
```

1 to 1 Functions, cont.

- Is square from \( \mathbb{Z} \) to \( \mathbb{Z} \) an example?
  - NO!
  - Because \( f_{\text{sq}}(-2) = 4 = f_{\text{sq}}(+2) \)!
- Is floor an example?
  - INCONCEIVABLE!!
- Is identity an example?
  - Unique at last!!

Increasing Functions

- A function \( f \) whose domain and co-domain are subsets of the set of real numbers is called increasing if \( f(x) \leq f(y) \) and strictly increasing if \( f(x) < f(y) \), whenever
  - \( x < y \) and
  - \( x \) and \( y \) are in the domain of \( f \).
- Is floor an example?
  - Yes
  - \( 1.5 < 1.7 \) and \( \text{floor}(1.5) = 1 = \text{floor}(1.7) \)
  - \( 1.2 < 2.2 \) and \( \text{floor}(1.2) = 1 < 2 = \text{floor}(2.2) \)
- Is square an example?
  - No
  - \( \text{square}(-2) = 4 > 1 = \text{square}(1) \) yet \( -2 < 1 \)

How is Increasing Useful?

- Most programs run longer with larger or more complex inputs.
- Consider the maze:
  - Larger maze may (in the worst case) take longer to get out.
  - Maze with more walls may (in the worst case) take longer to get out.
- Consider looking up a telephone number in the paper directory...
Cartesian Products and Functions

- A function with multiple arguments maps a Cartesian product of inputs to a codomain.
- Example:
  - `Math.min` maps (subset of) $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$
    ```java
    int minVal = Math.min(23, 99);
    ```
  - `Math.abs` maps (subset of) $\mathbb{Q}$ to $\mathbb{Q}^+$
    ```java
    int absVal = Math.abs(-23);
    ```

Quiz Check

- Is the following an increasing function?
  - $\mathbb{Z} \rightarrow \mathbb{Z}$  \( f(x) = x + 5 \)
  - $\mathbb{Z} \rightarrow \mathbb{Z}$  \( f(x) = 3x - 1 \)